1. step: Assuming that \( f(x) \) is differentiable at \( x = a \), from the picture we see:

\[
f(x) = f(a) + \Delta f \approx f(a) + f'(a)\Delta x
\]

**Linear approximation** of \( f(x) \) at point \( x \) around \( x = a \)

\[
f(x) = f(a) + f'(a)(x - a)
\]

**Remainder** \( R = f(x) - f(a) - f'(a)(x - a) \)

will determine magnitude of error

Let’s try to get better approximation of \( f(x) \).
Let’s assume that \( f(x) \) has all derivatives at \( x = a \).
Let’s assume there is a possible power series expansion of \( f(x) \) around \( a \).

\[
f(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + \cdots = \sum_{n=0}^{\infty} c_n (x - a)^n \quad \forall x \text{ around } a
\]

<table>
<thead>
<tr>
<th>( n )</th>
<th>( f^{(n)}(x) )</th>
<th>( f^{(n)}(a) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( f(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + \cdots )</td>
<td>( f(a) = c_0 )</td>
</tr>
<tr>
<td>1</td>
<td>( f'(x) = c_1 + 2c_2(x - a) + 3c_3(x - a)^2 + \cdots )</td>
<td>( f'(a) = c_1 )</td>
</tr>
<tr>
<td>2</td>
<td>( f''(x) = 2c_2 + 3 \cdot 2c_3(x - a) + 4 \cdot 3(x - a)^2 + \cdots )</td>
<td>( f''(a) = 2c_2 )</td>
</tr>
<tr>
<td>3</td>
<td>( f'''(x) = 3 \cdot 2c_3 + 4 \cdot 3 \cdot 2(x - a) + 5 \cdot 4 \cdot 3(x - a)^2 + \cdots )</td>
<td>( f'''(a) = 3 \cdot 2c_3 )</td>
</tr>
<tr>
<td>( n )</td>
<td>( f^{(n)}(x) = n(n - 1) \cdots 3 \cdot 2 \cdot 1c_n + (n + 1)n(n - 1) \cdots 2c_{n+1}(x - a) )</td>
<td>( f^{(n)}(a) = n! c_n )</td>
</tr>
</tbody>
</table>

**Taylor’s theorem**

If \( f(x) \) has derivatives of all orders in an open interval \( I \) containing \( a \), then for each \( x \) in \( I \),

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n = f(a) + f'(a)(x - a) + \cdots + \frac{f^{(n)}(a)}{n!} (x - a)^n + \frac{f^{(n+1)}(a)}{(n + 1)!} (x - a)^{n+1} + \cdots
\]

\[
= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x - a)^k + \sum_{k=n+1}^{\infty} \frac{f^{(k)}(a)}{k!} (x - a)^k
\]

\[
= T_n + R_n
\]

\[
T_n = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x - a)^k \text{ polynomial of } n^{th} \text{ order}
\]

\[
R_n(x) = \frac{f^{(n+1)}(z)}{(n + 1)!} (x - a)^{n+1} \text{ is remainder } R_n
\]

Taylor’s theorem says that there exists some value \( z \) between \( a \) and \( x \) for which:

\[
\sum_{k=n+1}^{\infty} \frac{f^{(k)}(a)}{k!} (x - a)^k \text{ can be replaced by } R_n(x) = \frac{f^{(n+1)}(z)}{(n + 1)!} (x - a)^{n+1}
\]
Approximating function by a polynomial function

So we are good to go. We can find value of \( f(x) \) around \( a \) by calculating \( T_n \), and then adding \( R_n \). Instead of adding infinite number of terms (how in the world??), we have finite number of terms. Beautiful.

A little problem: We know \( z \) exists, BUT we don’t know how to find \( z \). That’s why we use approximation:

By approximating \( f(x) \) with \( T_n \) we neglect \( R_n \). We can do it only if \( R_n \) is very small compared to \( T_n \).

So if we find maximum possible value for \( R_n \), and that value is small compared to \( T_n \), we found good approximation:

\[
f(x) \approx \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x - a)^k
\]

error is remainder \( R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!} (x - a)^{n+1} \)

\( z \) is the value between \( a \) and \( x \), that yields maximum \( R_n \)

---

**A Maclaurin series is a Taylor series expansion of a function about 0**

\[
\begin{align*}
\text{for } n = 0 & \quad f(x) = f(a) + f'(z)(x - a) \\
\text{ } & \quad f'(z) = \frac{f(x) - f(a)}{x - a} \rightarrow MEAN \text{ value theorem}
\end{align*}
\]

Problems:

• For what values of \( x \) can we expect a Taylor series to represent \( f(x) \)? → radius of convergence
• How accurately do Taylor polynomials approximate \( f(x) \)? → magnitude of the error \( R_n \)

**example 1: Find Taylor’s series for \( \sin x \) centered at \( a = 0 \) (McLaurin series).**

\[
\begin{align*}
f(x) &= \sin x \\
f'(x) &= \cos x \\
f''(x) &= -\sin x \\
f'''(x) &= -\cos x \\
f^{(4)}(x) &= \sin x
\end{align*}
\]

\( \text{this pattern will repeat} \)

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n = 0 + 1x + 0x^2 + \frac{-1}{3!} x^3 + 0x^4 + \ldots
\]

\[
= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n + 1)!}
\]

Signs alternate and the denominators get very big; factorials grow very fast.

Ratio test:

\[
\lim_{n \to \infty} \left| \frac{x^{2n+3}}{(2n + 3)!} \cdot \frac{(2n + 1)!}{x^{2n+1}} \right| = |x^2| \lim_{n \to \infty} \left| \frac{1}{(2n + 3)(2n + 2)} \right| = |x^2| \cdot 0
\]

This converges for any value of \( x \). The radius of convergence is infinity.
example 2: Find fifth order Taylor’s approximation for \( f(x) = \ln x \) centered at \( a = 1 \).

\[
\begin{align*}
f(x) &= \ln x \\
f'(x) &= \frac{1}{x} \\
f''(x) &= -\frac{1}{x^2} \\
f'''(x) &= \frac{2}{x^3} \\
f''''(x) &= -\frac{3!}{x^4} \\
f^{(5)}(x) &= \frac{4!}{x^5}
\end{align*}
\]

\[
f^{(n)}(x) = \frac{(-1)^{n+1} (n-1)!}{x^n} \rightarrow f^{(n)}(1) = (-1)^{n+1} (n-1)!
\]

\[
c_n = \frac{f^{(n)}(1)}{n!} = \frac{(-1)^{n+1}}{n}
\]

\[
\ln(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!} (x-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x-1)^n = (x-1) - \frac{1}{2} (x-1)^2 + \frac{1}{3} (x-1)^3 - \frac{1}{4} (x-1)^4 + \cdots \quad \text{for } |x-1| < 1
\]

\[
P_3(x) = (x-1) - \frac{1}{2} (x-1)^2 + \frac{1}{3} (x-1)^3 - \frac{1}{4} (x-1)^4 + \frac{1}{5} (x-1)^5
\]

example 2: Find Maclaurin series for \( f(x) = \ln (x+1) \)

\[
\begin{align*}
f(x) &= \ln(1+x) \\
f'(x) &= \frac{1}{1+x} \\
f''(x) &= -\frac{1}{(1+x)^2} \\
f'''(x) &= \frac{2}{(1+x)^3} \\
f''''(x) &= -\frac{3!}{(1+x)^4} \\
f^{(n)}(x) &= \frac{(-1)^{n+1} (n-1)!}{(1+x)^n}
\end{align*}
\]

\[
c_n = \frac{f^{(n)}(0)}{n!} = \frac{(-1)^{n+1}}{n}
\]

\[
\ln(1+x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \cdots \quad \text{for } |x| < 1 \quad -1 < x \leq 1
\]

Exact value of \( z \) is only a dream. If we knew it we would know exact expansion of a given function and it wouldn’t be approximation any more. Our goal is to find the bounds for \( f^{(n+1)}(z) \) so we can see how large remainder could be.

example: Use Taylor theorem to approximate \( \sin(0.1) \) by the third order polynom \( P_3(0.1) \) and determine the accuracy of the approximation.

\[
\sin x = x - \frac{x^3}{3!} + R_3(x) = x - \frac{x^3}{3!} + \frac{f^{(4)}(z)}{4!} x^4 \quad 0 < z < 0.1 \\
\sin(0.1) = 0.1 - \frac{(0.1)^3}{3!} = 0.0099833
\]

\[
0 < R_3(0.1) = \frac{\sin z}{4!} (0.1)^4 < \frac{0.0001}{4!} = 0.000004 \\
0.0099833 < \sin(0.1) < 0.0099833 + 0.000004 = 0.0099837
\]
example: Use Taylor theorem to approximate \( \ln (1.2) \) so error is less than 0.001

\[
R_n(x) = \frac{f^{(n+1)}(x)}{(n+1)!}(x-a)^{n+1} < 0.001
\]

1. Taylor theorem with \( a = 0 \) gives Maclaurin series of \( \ln (1+x) \) with \( x = 0.2 \)

In example 2 you saw that \((n+1)\)st derivate of \( f(x) = \ln(x+1) \) is given by

\[
f^{(n+1)}(x) = \frac{(-1)^{n+2} n!}{(1+x)^{n+1}} \rightarrow |R_n(0.2)| = \frac{|(-1)^{n+2} n!|}{(n+1)!} (0.2)^{n+1} = \frac{|(-1)^{n+2} n!|}{(n+1)(1+z)^{n+1}} (0.2)^{n+1}
\]

\( 0 < z < 0.2 \) maximum error would be for \( z = 0 \)

\[
\frac{|(-1)^{n+2} n!|}{(n+1)} (0.2)^{n+1} < 0.001 \quad \frac{(0.2)^{n+1}}{n+1} < 0.001
\]

Either trial and error or Wolframalpha \( n = 2.5 \rightarrow n = 3 \)

\( \ln(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} = \cdots \) with desired accuracy

2. Taylor theorem with \( a = 1 \) gives Taylor’s approximation for \( f(x) = \ln x \) with \( x = 1.2 \)

\[
R_n(x) = \frac{f^{(n+1)}(x)}{(n+1)!}(x-a)^{n+1} < 0.001
\]

In example 1 you saw that \((n+1)\)st derivate of \( f(x) = \ln(x) \) is given by

\[
f^{(n+1)}(x) = \frac{(-1)^{n+2} n!}{x^{n+1}} \rightarrow |R_n(1.2)| = \frac{|(-1)^{n+2} n!|}{(n+1)!} (1.2 - 1)^{n+1} = \frac{(0.2)^{n+1}}{(n+1) z^{n+1}}
\]

\( 1 < z < 1.2 \) maximum error would be for \( z = 1 \)

\[
\frac{|(-1)^{n+2} n!|}{(n+1)} (0.2)^{n+1} < 0.001 \quad \frac{(0.2)^{n+1}}{n+1} < 0.001 \quad \text{here we go again}
\]

---

**Convergence of Taylor Series**

If \( f(x) \) has derivatives of all orders in an open interval \( I \) containing \( a \), then for each \( x \) in \( I \),

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n
\]

holds if and only if

\[
\lim_{n \to \infty} R_n(x) = \lim_{n \to \infty} \frac{f^{(n+1)}(z)}{(n+1)!} (x - a)^{n+1} = 0 \quad \text{for} \quad \forall \, x \in I
\]
**Example:** Show that Maclaurin series for $f(x) = \sin x$ converges to $\sin x$ for all $x$.

You need to show that $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$ is true for all $x$.

For $R_n$ we need $f^{(n+1)}(z)$, and as $f^{(n+1)}(x)$ is either $\pm \sin x$ or $\pm \cos x$ we know that

maximum value of $|f^{(n+1)}(x)|$ is 1 for every real number $z$.

$$|R_n(x)| = \left| \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1} \right| \leq \frac{|x|^{n+1}}{(n+1)!} \to 0$$

From Taylor expansion $\frac{x^n}{n!} \leq e^x \forall x \geq 0$. Let $x = n$, $n^n e^{-n} \leq n!$ On the other hand, trivially $n! \leq n^n$

$$n^n e^{-n} \leq n! \leq n^n \rightarrow \frac{1}{n^n e^{-n}} \geq \frac{1}{n!} \geq \frac{e^n}{n^n} \geq \frac{x^n}{n^n} \geq \frac{x^n}{n^n}$$

$$\lim_{n \to \infty} \frac{|x|^{n+1}}{(n+1)!} \leq \lim_{n \to \infty} \frac{x^{n+1} e^{n+1}}{n!} = \lim_{n \to \infty} \left( \frac{ex}{n} \right)^{n+1} = 0 \quad \text{ex is fixed number no matter how big, so} \quad \frac{ex}{n} < 1 \text{ for } n \to \infty$$

Conclusion: $n!$ increase faster than any exponential function as $n \to \infty$, so $\lim_{n \to \infty} \frac{x^n}{n!} = 0$