BC CALCULUS REVIEW

Intermediate Value Theorem for Continuous Functions:
If \( f(x) \) is continuous on \([a, b]\) and \( k \) is any number between \( f(a) \) and \( f(b) \), then there is at least one number \( c \) between \( a \) and \( b \) such that \( f(c) = k \).

The Intermediate Value Theorem (IVT) is only an existence theorem. It does not explicitly tell us a number, just that a number exists.

**example:** Prove that function \( f(x) = x^2 \cos(2x) + 1 \) has a zero between 2 and 2.5.

\[
\begin{align*}
f(2) &\approx -1.614574 \\
f(2.5) &\approx 2.7728887
\end{align*}
\]

\( f \) is continuous on \([2, 2.5]\), \( f(2) < 0 \) and \( f(2.5) > 0 \), so \( f \) must have a zero between 2 and 2.5.

**Limit:** The limit is the \( y \)-value a function \( y(x) \) is getting close to not necessarily the function value itself.

**Right hand limit (RHL):** \( \lim_{x \to c^+} f(x) = L ; \) \( x > c \)

**Left hand limit (LHL):** \( \lim_{x \to c^-} f(x) = L ; \) \( x < c \)

**Limit of a function:** \( \lim_{x \to c} f(x) = L \) \iff \( \lim_{x \to c^-} f(x) = L \) and \( \lim_{x \to c^+} f(x) = L \)

**Continuous function at \( x = c \):** \( \lim_{x \to c^-} f(x) = f(c) \) \iff \( \forall x = c \) of domain

**Differentiable function at \( x = c \):** \( \lim_{x \to c^-} f'(x) = \lim_{x \to c^+} f'(x) \) for \( \forall x = c \) of domain

The derivative of a function \( y = f(x) \) at a fixed point \( a \)

\[
f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}
\]

The derivative of a function \( y = f(x) \)

\[
df \over dx = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}
\]

If a function is differentiable at \( x = c \), then it is continuous at \( x = c \).

The converse: "If a function is continuous at \( c \), then it is differentiable at \( c \)," is not true. **Differentiability implies continuity, continuity doesn’t imply differentiability.**

**visually:** left limit of the slope \neq \ right limit of the slope
Equation of the tangent line to a function \( y = f(x) \) at the point \( (a, f(a)) \)
\[
y - f(a) = f'(a)(x - a)
\]
Equation of the normal line to a function \( y = f(x) \) at the point \( (a, f(a)) \)
\[
y - f(a) = -\frac{1}{f'(a)} (x - a)
\]
Differentiation Formulas

\[
\begin{align*}
\frac{d}{dx} \left[ f(x) \pm g(x) \right] &= f'(x) \pm g'(x) \\
\frac{d}{dx} \left[ cf(x) \right] &= cf'(x) \\
\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] &= \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)} \\
\frac{d}{dx} \left[ \frac{1}{x} \right] &= -\frac{1}{x^2} \\
\frac{d}{dx} \left[ \ln x \right] &= \frac{1}{x} \\
\frac{d}{dx} \left[ \sin x \right] &= \cos x \\
\frac{d}{dx} \left[ \cos x \right] &= -\sin x \\
\frac{d}{dx} \left[ \tan x \right] &= \sec^2 x \\
\frac{d}{dx} \left[ \sec x \right] &= \sec x \tan x \\
\frac{d}{dx} \left[ \csc x \right] &= -\csc x \cot x \\
\frac{d}{dx} \left[ \cot x \right] &= -\csc^2 x
\end{align*}
\]

Chain rule for composite functions \( y = f(u(x)) \) \[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}
\]
\[
\left[ \frac{d}{dx} \left[ f(g(x)) \right] \right] = \frac{df}{dg} \cdot \frac{dg}{dx}
\]

<table>
<thead>
<tr>
<th>( \frac{d}{dx} \left[ u^n \right] = nu^{n-1} \frac{du}{dx} )</th>
<th>( \frac{d}{dx} \left[ e^u \right] = e^u \frac{du}{dx} )</th>
<th>( \frac{d}{dx} \left[ \ln u \right] = \frac{1}{u} \frac{du}{dx} )</th>
</tr>
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<tr>
<td>( \frac{d}{dx} \left[ \sin u \right] = \cos u \frac{du}{dx} )</td>
<td>( \frac{d}{dx} \left[ \cos u \right] = -\sin u \frac{du}{dx} )</td>
<td>( \frac{d}{dx} \left[ \tan u \right] = \sec^2 u \frac{du}{dx} )</td>
</tr>
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<td>( \frac{d}{dx} \left[ \sec u \right] = \sec u \tan u \frac{du}{dx} )</td>
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</tr>
</tbody>
</table>

\[
\begin{align*}
y &= \sin(x^2) \\
y &= \sin u \\
u &= x^2 \\
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} = (\cos u) (2x) \\
\frac{dy}{dx} &= 2x \cos (x^2)
\end{align*}
\]

\[
\begin{align*}
y &= \sin^2 x \\
y &= u^2 \\
u &= \sin x \\
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} = (2u) (\cos x) \\
\frac{dy}{dx} &= 2 \sin x \cos x
\end{align*}
\]

\[
\begin{align*}
f(x) &= \frac{1}{\sqrt{x^2 + 1}} \\
f(x) &= u^{-\frac{1}{2}} \\
u &= x^2 + x + 1 \\
f'(x) &= -\frac{1}{3} (x^2 + x + 1)^{-\frac{4}{3}} (2x + 1) \\
f'(x) &= -\frac{2x + 1}{3\sqrt[3]{(x^2 + x + 1)^4}}
\end{align*}
\]
If \( f(x) \) represents a quantity, then \( f'(x) = \frac{df}{dx} \) represents the instantaneous rate of change of that function.

### KINEMATICS SUMMARY

**Displacement, position: \( s(t) \)**

- **Average velocity:** \( v(t) = \frac{s_2 - s_1}{\Delta t} \)
- **Instantaneous velocity:** \( v(t) = \frac{ds}{dt} \)
- **Instantaneous speed:** \( |v(t)| \)

**Average acceleration:** \( a(t) = \frac{v_2 - v_1}{\Delta t} \)

**Instantaneous acceleration:** \( a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2} \)

**Magnitude of acceleration:** \( |a(t)| \)

1. if \( \frac{ds}{dt} > 0 \) → \( v \) is + and \( |v| \) is +
   - **1a.** if \( v > 0 \) and \( \frac{dv}{dt} > 0 \) → \( v \) ↑ and \( |v| \) ↑

2. if \( \frac{ds}{dt} < 0 \) → \( v \) is − and \( |v| \) is +
   - **2a.** if \( v < 0 \) and \( \frac{dv}{dt} > 0 \) → \( v \) ↑ and \( |v| \) ↓

3. if \( \frac{ds}{dt} < 0 \) → \( v \) is − and \( |v| \) is −
   - **2b.** if \( v < 0 \) and \( \frac{dv}{dt} < 0 \) → \( v \) ↓ and \( |v| \) ↓

### IMPLICIT DERIVATION: use when functions are defined implicitly: \( x^2 + y^2 = 25 \)

\[
x^3 + y^3 = 6xy, \\
x^2 + 3y^2y' = 6xy' + 6y \\
x^2 + y^2y' = 2xy' + 2y \\
y' = \frac{2y - x^2}{y^2 - 2x}
\]

**Chain rule:** \( u = y^3 \)

\[
du = \frac{dy}{dx} \frac{du}{dy} = 3y^2 \frac{dy}{dx} = 3y^2y' \\
\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}
\]

**2a.** if \( v > 0 \) and \( \frac{dv}{dt} > 0 \) → \( v \) ↑ and \( |v| \) ↑

\[
2y = x^2 + \sin y \\
\frac{dy}{dx} = 2x + \cos y \\
\frac{dy}{dx} = \frac{2x}{2 - \cos y} \\
\frac{dy}{dx} = \frac{2\sin y}{2 - \cos y}
\]

**2b.** if \( v < 0 \) and \( \frac{dv}{dt} < 0 \) → \( v \) ↓ and \( |v| \) ↓

\[
2y = x^2 + \sin y \\
\frac{dy}{dx} = 2x + \cos y \\
\frac{dy}{dx} = \frac{2x}{2 - \cos y} \\
\frac{dy}{dx} = \frac{2\sin y}{2 - \cos y}
\]

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\frac{dy}{dx} = \frac{2x}{2 - \cos y} \\
\frac{dy}{dx} = \frac{2\sin y}{2 - \cos y}
\]
LOGARITHMIC DIFFERENTIATION:

$f(x)^{g(x)}$ like $x^x$, $x^{\sin x}$ or a crazy product, quotient

1. Take natural logarithms of both sides of an equation $y = f(x)$.
2. Differentiate implicitly with respect to $x$
3. Solve the resulting equation for $y'$.

$$y = x^{\sin x}$$
$$\ln y = (\sin x) \ln x$$
$$\frac{1}{y} y' = (\cos x) \ln x + \frac{\sin x}{x}$$
$$y' = (\ln x)x^{\sin x} \cos x + (\sin x) x^{\sin x-1}$$

**Derivative of inverse function**

$g(x) = f^{-1}(x)$ ⇒ $f[g(x)] = x$ ⇒ $\frac{df}{dx} \frac{dg}{dx} = 1$ ⇒ $g'(x) = \frac{1}{f'(g(x))}$

$$f(x) = 2x + \cos x, \text{ find } (f^{-1})'(1)$$
$$f'[g(x)] = x \Rightarrow f'[g(x)]g'(x) = 1 \Rightarrow$$
$$f'[g(1)]g'(1) = 1$$
$$f(0) = 1 \Rightarrow g(1) = f^{-1}(1) = 0$$
$$g'(1) = \frac{1}{f'(0)} = \frac{1}{2 - \sin 0} \Rightarrow g'(1) = \frac{1}{2}$$

**Intermediate Value Theorem for Derivatives:**

If $a$ and $b$ are any two points in an interval on which $f$ is differentiable, then $f'$ takes on every value between $f'(a)$ and $f'(b)$.

Between $a$ and $b$, $f'$ must take on every value between $\frac{1}{2}$ and 3.

The Intermediate Value Theorem (IVT) is only an existence theorem. It does not explicitly tell us a number, just that a number exists.
Methods for finding limits

\[
\lim_{x \to \pm \infty} \frac{P(x)}{Q(x)} \quad \text{Remember} \quad \lim_{x \to \pm \infty} \frac{1}{x} = 0
\]

\[
\lim_{x \to \infty} \frac{x^3 + x - 6}{x^2 - 4x + 3} = \lim_{x \to \infty} \frac{x^2(x + 1/x - 6/x^2)}{x^2(1 - 4/x + 3/x^2)} = \lim_{x \to \infty} \frac{x}{1} = \infty
\]

\[
\lim_{x \to \infty} \frac{x^2 - 4x + 3}{x^3 + x - 6} = \lim_{x \to \infty} \frac{x^2(1 - 4/x + 3/x^2)}{x^2(x + 1/x - 6/x^2)} = \lim_{x \to \infty} \frac{1}{x} = 0
\]

\[\text{Use} \quad \lim_{x \to 0} \frac{\sin x}{x} = 1\]

\[
\lim_{x \to 0} \frac{\tan 3x}{\sin 2x} = \lim_{x \to 0} \frac{\cos 3x}{\sin 3x} \cdot \frac{\sin 2x}{\sin 2x} = \lim_{x \to 0} \left( \frac{\sin 3x}{1} \cdot \frac{1}{\cos 3x} \cdot \frac{1}{\sin 2x} \right) = \frac{3}{2} \left( \lim_{x \to 0} \frac{\sin 3x}{3x} \right) \left( \lim_{x \to 0} \frac{1}{\cos 3x} \right) \left( \lim_{x \to 0} \frac{2x}{\sin 2x} \right) = \frac{3}{2} \cdot 1 \cdot 1 \cdot 1 = \frac{3}{2}
\]

\[
\lim_{x \to 0} \frac{1}{x} \sin \left( \frac{1}{x} \right) = \left( \text{substitution} \ y = \frac{1}{x} \right) = \lim_{y \to 0} \frac{2x}{\sin y} = \lim_{y \to 0} \frac{1}{\sin y} = \frac{1}{1} = 1
\]

Recognizing a given limit \( \left( \frac{0}{0} \right) \) as a derivative (!!!!!):

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \quad \text{or} \quad f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}
\]

\[
\lim_{h \to 0} \frac{\sqrt{8 + h} - 2}{h} = \frac{d}{dx} \sqrt{x} \bigg|_{x=8} = \frac{1}{3} x^{-2/3} \bigg|_{x=8} = \frac{1}{12}
\]

\[
\lim_{x \to \pi/3} \frac{\cos x - \frac{1}{2}}{x - \frac{\pi}{3}} = \lim_{h \to 0} \frac{\cos x - \cos \frac{\pi}{3}}{x - \frac{\pi}{3}} = \frac{d}{dx} \cos x \bigg|_{x=\pi/3} = -\sin x \bigg|_{x=\pi/3} = -\frac{\sqrt{3}}{2}
\]
4. L'Hopital Rule

a) Indeterminate forms: \( \frac{0}{0} \) and \( \pm \infty \pm \infty \) : use L'Hopital rule directly

\[
\lim_{x \to 0} \frac{xe^{3x} - x}{1 - \cos(2x)} = \lim_{x \to 0} \frac{3xe^{3x} + e^{3x} - 1}{2 \sin(2x)} = \frac{0}{0} = \lim_{x \to 0} \frac{9xe^{3x} + 3e^{3x} + 3e^{3x}}{4 \cos(2x)} = \frac{6}{4} = \frac{3}{2}
\]

\[
\lim_{x \to \infty} \frac{x}{(\ln x)^3} = \lim_{x \to \infty} \frac{1}{3(\ln x)^2} \left( \frac{x}{x} \right) = \lim_{x \to \infty} \frac{x}{3(\ln x)^2} = \left( \frac{\infty}{\infty} \right)
\]

\[
= \lim_{x \to \infty} \frac{1}{6 \ln x (1/x)} = \lim_{x \to \infty} \frac{x}{6 \ln x} = \left( \frac{\infty}{\infty} \right) = \lim_{x \to \infty} \frac{1}{6(1/x)} = \lim_{x \to \infty} \frac{x}{6} = \infty
\]

b) Indeterminate form: \( 0 \cdot \infty \) use algebra to convert the expression to a fraction \( (0 \cdot \infty) = 0 \cdot \frac{1}{0} = \frac{0}{0} \)

\[
\lim_{x \to 0} (\arcsin x) (\cot x) = (0 \cdot \infty) = \lim_{x \to 0} \frac{\cos x \arcsin x}{\sin x} = \left( \frac{0}{0} \right) = \lim_{x \to 0} \frac{-\sin x \arcsin x + \frac{\cos x}{\sqrt{1 - x^2}}}{\cos x} = 1
\]

\[
\lim_{x \to \infty} x^x = (\infty \cdot 0) = \lim_{x \to \infty} \left( \frac{e^{x/2} - 1}{(1/x)} \right) = \left( \frac{0}{0} \right) = \lim_{x \to \infty} \frac{e^{1/x}(-1/x^2)}{-1/x^2} = \lim_{x \to \infty} \left( e^{1/x} \right) = e^0 = 1
\]

c) Indeterminate form: \( \infty - \infty \) convert the expression into \( \frac{0}{0} \) or \( \pm \infty \pm \infty \)

\[
\lim_{x \to 1} \frac{x - 1}{\ln x} = \lim_{x \to 1} \frac{x \ln x - x + 1}{(x - 1)\ln x} = \lim_{x \to 1} \frac{\ln x + 1 - 1}{\ln x + x - 1} = \lim_{x \to 1} \frac{x \ln x}{x \ln x + (x - 1)} = \lim_{x \to 1} \frac{\ln x + 1}{\ln x + 2} = \frac{1}{2}
\]

d) Indeterminate forms: \( 1^\infty \) \( 0^0 \) \( \infty^0 \)

1. \( y = f(x)g(x) \)

\[
\lim_{x \to a^+} x^x = (0^0) \quad \quad \lim_{x \to a^+} x^x = (0^0) \quad \quad y = x^x
\]

2. \( \ln y = g(x) \ln f(x) \)

\[
\lim_{x \to 0^+} \ln x = (0 \cdot -\infty) = \lim_{x \to 0^+} \frac{\ln x}{x} = \left( \frac{\infty}{\infty} \right) = \lim_{x \to 0^+} \frac{1}{x} = \lim_{x \to 0^+} (-x) = 0
\]

3. \( \lim_{x \to a} (g(x) \ln f(x)) = L \)

\[
\lim_{x \to 0^+} x \ln x = (0 \cdot -\infty) = \lim_{x \to 0^+} \frac{\ln x}{x} = \left( \frac{\infty}{\infty} \right) = \lim_{x \to 0^+} \frac{1}{x} = \lim_{x \to 0^+} (-x) = 0
\]

4. \( \lim_{x \to 0} e^x = e^0 = 1 \)

\[
\lim_{x \to 0^+} x^x = (0^0) \quad \lim_{x \to 0^+} x^x = (0^0) \quad y = x^x
\]
Absolute Extrema
1. \( x = c \) is an absolute maximum of \( f(x) \) if
\( f(c) \geq f(x) \) for all \( x \) in the domain.
2. \( x = c \) is an absolute minimum of \( f(x) \)
   if \( f(c) \leq f(x) \) for all \( x \) in the domain.

Relative (local) Extrema
1. \( x = c \) is a relative/local maximum of \( f(x) \) if
\( f(c) \geq f(x) \) for all \( x \) in some open interval around \( c \).
2. \( x = c \) is a relative/local minimum of \( f(x) \) if
\( f(c) \leq f(x) \) for all \( x \) in some open interval around \( c \).

If the function isn’t continuous on the interval then the function may or may not have absolute/local extrema at the point of discontinuity.

Critical Point:
\( x = c \) is a critical point of \( f(x) \) provided that either
1. \( f'(c) = 0 \)
   or
2. \( f''(c) \) does not exist.

Note: A critical value can occur at a discontinuity, as long as \( f \) is defined at \( x = c \)
Note: A possible inflection value can occur at a discontinuity, as long as \( f \) is defined at \( x = c \)

Possible inflection value:
\( x = c \) is possible inflection value of \( f(x) \) provided that either
1. \( f''(x) = 0 \)
   or
2. \( f''(x) = DNE \)

These values are essentially critical values of \( f'(x) \)

CURVE SKETCHING

INCREASING/DECREASING TEST
- If \( f'(x) > 0 \) on an interval, then \( f \) is increasing on that interval.
- If \( f'(x) < 0 \) on an interval, then \( f \) is decreasing on that interval.

CONCAVITY TEST
- If \( f''(x) > 0 \) on an interval, then \( f \) is concave up on that interval.
- If \( f''(x) < 0 \) on an interval, then \( f \) is concave down on that interval.

Inflection point:
- \( f''(x) = 0 \)
- \( f''(x) \) changes from either
  + to − OR − to +

Local minimum:
- \( f''(x) > 0 \)
- \( f''(x) \) changes from − to +
- \( f(x) > f(c) \) around \( c \)

Local maximum:
- \( f''(x) < 0 \)
- \( f''(x) \) changes from + to −
- \( f(x) < f(c) \) around \( c \)
CLOSED INTERVAL TEST for ABSOLUTE EXTREMA

For a continuous function \( f(x) \) on a closed interval \([a, b]\):
1) Find all critical points of \( f(x) \) \[ f'(x) = 0 \& f'(x) = \text{DNE} \]
2) Find all extrema \( [f'(x) = 0] \)
   check for min/max
   a) check \( f(x) \) on both sides of extrema point \( x = c \) \[ a < c \& b > c \]
      if \( f(a) \& f(b) < f(c) \) \( \Rightarrow \) maximum \( @ x = c \)
      if \( f(a) \& f(b) > f(c) \) \( \Rightarrow \) minimum \( @ x = c \)
   or
   b) check \( f'(x) \) on both sides of extrema point \( x = c \)
      if \( f'(a) \text{is positive} \& f'(b) \text{is negative} \Rightarrow \) maximum \( @ x = c \)
      if \( f'(a) \text{is negative} \& f'(b) \text{is positive} \Rightarrow \) minimum \( @ x = c \)
   or
   c) check \( f''(x) @ \text{extrema point } x = c \)
      if \( f''(c) \text{is negative} \Rightarrow f'(c) \text{is decreasing [form "+" to "]" } \Rightarrow \) maximum \( @ x = c \)
      if \( f''(c) \text{is positive} \Rightarrow f'(c) \text{is increasing [form "]" to "+"] } \Rightarrow \) minimum \( @ x = c \)
3) for absolute global extrema check end points !!!

Note: You may find critical values of \( f \) that are not in the open interval \((a, b)\).
While these will certainly be critical values of \( f \), they are not included in the test if they are not in the open interval \((a, b)\).

---

Rolle's Theorem:

If \( f(x) \) is
1) continuous on \([a, b]\)
2) differentiable on \((a, b)\)
3) \( f(a) = f(b) \)

\[ \exists \, c \in (a, b) \text{ such that } f'(c) = 0 \]

(that means that there exists at least one)

Mean Value Theorem for Derivatives:

If \( f(x) \) is
1) continuous on \([a, b]\)
2) differentiable on \((a, b)\),

then there exists a number \( c \) in \((a, b)\) such that

\[ f'(c) = \frac{f(b) - f(a)}{b - a} \]

Geometric Interpretation: Under the given conditions, there is a point in the open interval where the tangent to the curve is the same as the slope of the line joining the endpoints.

Application: Under the given conditions, there is a point in the open interval where the instantaneous rate of change is the same as the average rate of change on the interval (very important). If the function is a position function, then there is a point in the open interval where the instantaneous velocity is the same as the average velocity on the interval.

\[ \frac{f(b) - f(a)}{b - a} \text{ is the average rate of change of the function } f(x) \text{ on the interval } [a, b] \]
OPTIMIZATION PROBLEMS

- involve setting up an equation to be optimized and constraint.
- solve constraint for one of two variables and plug it into first equation.
- find stationary points for optimal solution (min/max)
- the optimal solution may occur at and end point of the domain as well as at a stationary point.

We want to construct a box with a square base and we only have 10 square meters of material to use in construction of the box. Assuming that all the material is used in the construction process determine the maximum volume that the box can have.

<table>
<thead>
<tr>
<th>Constraint eq.</th>
<th>Optimization eq.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2a^2 + 4ab = 10$</td>
<td>$V = a^2b$</td>
</tr>
</tbody>
</table>

\[ V = a^2b = \frac{5a}{2} - a^3/2 \]

\[ \frac{dV}{da} = \frac{5}{2} - \frac{3a^2}{2} = 0 \]

\[ a = \sqrt[3]{\frac{5}{3}} \quad b = \sqrt[3]{\frac{5}{3}} \]

A cylindrical aluminum soda can is to hold 300 cm$^3$ of tasty beverage. What are the dimensions of the can that will minimize the amount of aluminum used?

<table>
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<td>$(\pi r^2)h = 300$</td>
<td>$A = (2\pi r)h + 2(\pi r^2)$</td>
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\[ A = \frac{600}{r} + 2\pi r^2 \]

\[ \frac{dA}{dr} = -\frac{600}{r^2} + 4\pi r \]

\[ \frac{dA}{dr} = 0 \]

\[ r = 3.628 \text{ cm} \quad h = 7.287 \text{ cm} \]

\[ A_{\text{min}} = 248.812 \text{ cm}^3 \]

\[ A_{\text{min}} \approx 249 \text{ cm}^3 \]

RELATED RATES

Objectives: Identify a mathematical relationship between quantities that are each changing.

- Draw a picture (sketch).
- Write down known information.
- Write down what you are looking for.
- Write an equation to relate the variables.
- Differentiate both sides with respect to $t$ using chain rule.

Evaluate

Suppose that the radius of a sphere is increasing 0.1 cm/sec. At what rate is the sphere growing when $r = 10$ cm?

\[ \frac{dr}{dt} = 0.1 \text{ cm/sec} \quad \left( \frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} \right) \]

\[ V = \frac{4}{3} \pi r^3 \]

\[ \frac{dV}{dr} = 4\pi r^2 \text{dr} \]

\[ \frac{dV}{dt} \bigg|_{r=10} = 40\pi \text{ cm}^3/\text{sec} \]

Air is being blown into a sphere at the rate of 6 cm$^3$/s. How fast is the radius changing when the radius of the sphere is 2 cm?

\[ \frac{dV}{dt} = \frac{6 \text{ cm}^3}{s} \quad \left( \frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} \right) \]

\[ V = \frac{4}{3} \pi r^3 \]

\[ \frac{dV}{dt} = 4\pi r^2 \text{dr} \]

\[ \frac{at \ r = 2 \text{ cm}}{\text{dr}} = 6 = 4\pi(2)^2 \frac{dr}{dt} \]

\[ \frac{dr}{dt} \bigg|_{r=2} = \frac{3}{8\pi} \text{ cm/sec} \]
Balloon is rising straight up. Angle $\theta$ is changing $0.14 \text{ rad/min}$. How fast is balloon rising when $\theta = \pi/4$?

$$\frac{d\theta}{dt} = 0.14 \text{ rad/ min} \quad \frac{dh}{dt} = \text{?}$$

$h = 500 \tan \theta$

$$\frac{dh}{dt} = 500 \sec^2 \theta \frac{d\theta}{dt}$$

$$\left. \frac{dh}{dt} \right|_{\theta = \pi/4} = 500 \left( \sec^2 \frac{\pi}{4} \right)^2 (0.14)$$

$$\left. \frac{dh}{dt} \right|_{\theta = \pi/4} = 140 \text{ m/min}$$

Car A is traveling west at 50 mi/h and car B is traveling north at 60 mi/h. Both are headed for the intersection of the two roads.

At what rate are the cars approaching each other when car A is 0.3 mi and car B is 0.4 mi from the intersection?

$$\frac{dx}{dt} = -50 \text{ mi/h} \quad \frac{dy}{dt} = -60 \text{ mi/h}$$

$$\frac{dz}{dt} = ?$$

$$x^2 = x^2 + y^2$$

$$2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$$

When $x = 0.3 \text{ mi}$ & $y = 0.4 \text{ mi}$, $z = 0.5 \text{ mi}$.

$$\frac{dz}{dt} = -78 \text{ mi/h}$$

---

### Linear approximation / Tangent line approximation of $f(x)$ at point $x$ around $a$

$$f(x) = f(a) + f'(a)(x - a)$$

**STEPS**

1. Find the equation of the tangent line $y(x)$ at $(a, f(a))$
   $$y(x) = f(a) + f'(a)(x - a)$$

2. For some $x = c$ around $a$: $f(c) \approx y(c)$
   $$f(c) \approx f(a) + f'(a)(c - a)$$

---

3. To determine if $y(c)$ is an over-approximation or an under approximation find concavity of $f(x)$ at $x = a$

   a. If $f''(a) < 0$, $f(x)$ is concave down around $a$, tangent line is above $f(x)$ and $y(c)$ is an over-approximation

   b. If $f''(a) > 0$, $f(x)$ is concave up around $a$, tangent line is below $f(x)$ and $y(c)$ is an under-approximation

**example:**

Suppose that we don't have a formula for $g(x)$ but we know that $g(3) = -3$ and $g'(x) = x^2 + 7$ for all $x$

(a) Use a linear approximation to estimate $g(2.9)$ and $g(3.1)$.

(b) Are your estimates in part (a) too large or too small? Explain

$$g(x) = y(x) = g(a) + g'(a)(x - a)$$

$$g(2.9) = -(3 + 16(-0.1)) = -4.6$$

$$g(3.1) = -(3 + 16(0.1)) = -2.4$$

$g'(3) > 0$ $g$ increases so the linear approx will be less than the actual value since the linear approximation will underestimate the increasing slope.
example:
Find the linearization of the function \( f(x) = \sqrt{x} + 3 \) at \( a = 1 \), and use it to approximate the numbers \( \sqrt{3.98} \) and \( \sqrt{4.05} \).

Are these approximations overestimates or underestimates?

\[
f'(x) = \frac{1}{2\sqrt{x} + 3}
\]

\[
f(x) \approx f(1) + [f'(x)]_{x=1}(x - 1)
\]

\[
f(x) \approx 2 + \frac{1}{4}(x - 1) = \frac{7}{4} + \frac{x}{4}
\]

linear approximation: \( \sqrt{x + 3} \approx \frac{7}{4} + \frac{x}{4} \) (when \( x \) is near 1)

\[
\sqrt{3.98} \approx \frac{7}{4} + \frac{0.98}{4} = 1.995
\]

\[
\sqrt{4.05} \approx \frac{7}{4} + \frac{1.05}{4} = 2.0125
\]

\( f''(1) < 0 \) \( \rightarrow \) \( f(x) \) is concave down around \( x = 1 \)

Tangent line is above the curve, so estimated values are overestimation

---

**Differentials:**

Let \( f(x) \) be a differentiable function. The differential \( dy \) is:

\[
dy = \frac{dy}{dx} \, dx
\]

\[
dy = f'(x) \, dx
\]
INTEGRATION

1. **Antiderivative** of \( f(x) \) is a function, \( F(x) \), such that \( F'(x) = f(x) \).

2. **Indefinite integral**: \( \int f(x) \, dx = F(x) + C \) where \( F(x) \) is an antiderivative of \( f(x) \).

3. **Definite integral**

   \[
   \Delta x = \frac{b - a}{n}, \quad \text{and} \quad x_i = a + (n - 1)\Delta x
   \]

   \[
   \text{Riemann sum of } f(x) \text{ on } [a, b] \text{ is } \sum_{i=1}^{n} f(x_i)\Delta x
   \]

   In limit as \( n \to \infty \)

   \[
   \frac{f(x_i)}{\Delta x} \to f(x)
   \]

   \[
   \Delta x \to dx
   \]

   \[
   \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i)\Delta x = \int_{a}^{b} f(x) \, dx = F(b) - F(a)
   \]

**Fundamental Theorem of Calculus (FToC1):**

\[
\text{definite integral } \int_{a}^{b} f(x) \, dx = F(b) - F(a)
\]

If \( f'(x) = 4\sin^2(2x) \) and \( f(2) = -2 \) find \((a)\) and integral equation for \( f(x) \)

\[\begin{align*}
   a) \quad f(x) &= f(2) + \int_{2}^{x} 4\sin^2(2x) \, dx = f(2) + 2 \int_{2}^{x} (1 - \cos 4x) \, dx = f(2) + 2x \bigg|_{2}^{x} - \frac{1}{2} \sin 4x \bigg|_{2}^{x} \\
   &= f(x) = 2x - 6 - \sin 4x + \frac{1}{2} \sin(8) \\
   b) \quad f(3) &= f(2) + \int_{2}^{3} 4\sin^2(2x) \, dx = f(2) + 2 \int_{2}^{3} (1 - \cos 4x) \, dx = -2 + 2x \bigg|_{2}^{3} - \frac{1}{2} \sin 4x \bigg|_{2}^{3} \\
   &= f(3) = \frac{1}{2} \sin 8 - \frac{1}{2} \sin 12 \\
   c) \quad f(-2) &= -2 + \int_{2}^{-2} 4\sin^2(2x) \, dx = -2 + 2 \int_{2}^{-2} (1 - \cos 4x) \, dx = -2 + 2x \bigg|_{2}^{-2} - \frac{1}{2} \sin 4x \bigg|_{2}^{-2} \\
   &= f(-2) = \sin(8) - 10
\end{align*}\]

area as definite integral — AREA IS ALWAYS POSITIVE IN MATH

1. If we are integrating by hand, we must decide if and where the graph crosses the \( x \)-axis, then split up our interval, manually making negative regions positive.
2. If the **calculator** is permitted, then evaluate

   2. Calculator: \( \int_{a}^{b} |f(x)| \, dx \)

   In velocity – time graph, net accumulation of integral is **displacement** the area (absolute value) is **distance** !!!!
**Difference between the Value of a Definite Integral and Total Area**

1. Find the mean value of the function \( f(x) = \sin x \) over interval \([0, 2\pi]\)

\[
\frac{1}{2\pi} \int_{0}^{2\pi} \sin x \, dx = \frac{1}{2\pi} \left[ -\cos x \right]_{0}^{2\pi} = 0
\]

2. Find total area \( f(x) = \sin x \) over interval \([0, 2\pi]\)

\[
\int_{0}^{2\pi} |\sin x| \, dx = \int_{0}^{\pi} \sin x \, dx - \int_{\pi}^{2\pi} \sin x \, dx = -[-2] + [2] = 4
\]

Definite integral is net accumulation

---

**Fundamental Theorem of Calculus 2 (FToC2)**

\( f(x) \) is continuous on \([a, b]\). Then

\[
F(x) = \int_{a}^{x} f(t) \, dt
\]

is also continuous on \([a, b]\), and

\[
F'(x) = \frac{d}{dx} \int_{a}^{x} f(t) \, dt = f(x)
\]

FToC 2, the most general form

\[
\frac{d}{dx} \left[ \int_{h(x)}^{g(x)} f(t) \, dt \right] = f(g(x)) \cdot g'(x) - f(h(x)) \cdot h'(x)
\]

definite integrals properties

\[
\begin{align*}
\int_{a}^{b} f(x) \, dx &= \int_{a}^{b} f(t) \, dt \\
\int_{a}^{b} f(x) \, dx &= 0 \\
\int_{a}^{b} f(x) \, dx &= -\int_{b}^{a} f(x) \, dx \\
\int_{a}^{b} g(x) \, dx &= c \int_{a}^{b} f(x) \, dx
\end{align*}
\]

\[
\left| \int_{a}^{b} f(x) \, dx \right| \leq \int_{a}^{b} |f(x)| \, dx
\]
Common derivatives and integrals

<table>
<thead>
<tr>
<th>Derivative</th>
<th>Integral</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{d}{dx}</td>
<td>x</td>
</tr>
<tr>
<td>$\frac{d}{dx} x^n = nx^{n-1}$</td>
<td>$\int x^n , dx = \frac{x^{n+1}}{n+1} + C$</td>
</tr>
<tr>
<td>$\frac{d}{dx} e^x = e^x$</td>
<td>$\int e^x , dx = e^x + C$</td>
</tr>
<tr>
<td>$\frac{d}{dx} a^x = a^x \ln a$</td>
<td>$\int a^x , dx = \frac{a^x}{\ln a} + C$</td>
</tr>
<tr>
<td>$\frac{d}{dx} \ln x = \frac{1}{x}$</td>
<td>$\int \frac{1}{x} , dx = \ln</td>
</tr>
<tr>
<td>$\frac{d}{dx} \sin x = \cos x$</td>
<td>$\int \cos x , dx = \sin x + C$</td>
</tr>
<tr>
<td>$\frac{d}{dx} \cos x = -\sin x$</td>
<td>$\int -\sin x , dx = -\cos x + C$</td>
</tr>
<tr>
<td>$\frac{d}{dx} \tan x = \sec^2 x$</td>
<td>$\int \sec^2 x , dx = \tan x + C$</td>
</tr>
<tr>
<td>$\frac{d}{dx} \cot x = -\csc^2 x$</td>
<td>$\int -\csc^2 x , dx = -\cot x + C$</td>
</tr>
<tr>
<td>$\frac{d}{dx} \csc x = -\csc x \cot x$</td>
<td>$\int -\csc x \cot x , dx = -\csc x + C$</td>
</tr>
<tr>
<td>$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$</td>
<td>$\int \frac{1}{\sqrt{1-x^2}} , dx = \arcsin x + C$</td>
</tr>
<tr>
<td>$\frac{d}{dx} \arccos x = -\frac{1}{\sqrt{1-x^2}}$</td>
<td>$\int -\frac{1}{\sqrt{1-x^2}} , dx = -\arccos x + C$</td>
</tr>
<tr>
<td>$\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$</td>
<td>$\int \frac{1}{1+x^2} , dx = \arctan x + C$</td>
</tr>
<tr>
<td>$\frac{d}{dx} \arccot x = -\frac{1}{1+x^2}$</td>
<td>$\int -\frac{1}{1+x^2} , dx = -\arccot x + C$</td>
</tr>
<tr>
<td>$\frac{d}{dx} \text{arcsec} x = \frac{1}{x\sqrt{x^2-1}}$</td>
<td>$\int \frac{1}{x\sqrt{x^2-1}} , dx = \text{arcsec} x + C$</td>
</tr>
<tr>
<td>$\frac{d}{dx} \text{arccsc} x = -\frac{1}{x\sqrt{x^2-1}}$</td>
<td>$\int -\frac{1}{x\sqrt{x^2-1}} , dx = -\text{arcsec} x + C$</td>
</tr>
</tbody>
</table>

Displacement (or change in position) = $\int_{a}^{b} v(t) \, dt$

Total Distance Traveled = $\int_{a}^{b} |v(t)| \, dt$. 
Standard Integration Techniques

**u Substitution:** The substitution \( u = g(x) \) & \( du = g'(x)dx \) will convert

\[
\int_a^b f[g(x)]g'(x)dx = \int_{g(a)}^{g(b)} f(u)du
\]
don’t forget to change limits of integration to new variable

\[
\int_1^2 5x^2 \cos x^3 \, dx = \left( \frac{u = x^3}{du = 3x^2 \, dx} \right) = \frac{5}{3} \int_1^8 \cos u \, du = \frac{5}{3} \sin u \bigg|_1^8 = \frac{5}{3}(\sin 8 - \sin 1)
\]

<table>
<thead>
<tr>
<th>function contains</th>
<th>substitution</th>
<th>example</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sqrt{f(x)} )</td>
<td>( u = f(x) ) ( du = u' , dx )</td>
<td>( \left[ 1/\sqrt{(-2x + 3)} \right] dx = (-\frac{1}{2})du/\sqrt{u} )</td>
</tr>
<tr>
<td>( \ln x )</td>
<td>( u = \ln x ) ( du = (1/x) , dx )</td>
<td>( (\ln^2 x / x) , dx = u^2 , du )</td>
</tr>
<tr>
<td>( x^2 + a^2 ) or ( \sqrt{x^2 + a^2} )</td>
<td>( x = a \tan \theta ) ( dx = a \sec^2 \theta , d\theta )</td>
<td>( a^2 \tan^2 \theta + a^2 = a^2 \sec^2 \theta )</td>
</tr>
<tr>
<td>( \sqrt{a^2 - x^2} )</td>
<td>( x = a \sin \theta ) ( dx = a \cos \theta , d\theta )</td>
<td>( a^2 - a^2 \sin^2 \theta = a^2 \cos^2 \theta )</td>
</tr>
<tr>
<td>( \sqrt{x^2 - a^2} )</td>
<td>( x = a \sec \theta ) ( dx = a \sec \theta \tan \theta , d\theta )</td>
<td>( a^2 \sec^2 \theta - a^2 = a^2 \tan^2 \theta )</td>
</tr>
</tbody>
</table>

\[
\frac{16}{x^2 \sqrt{4 - 9x^2}} \, dx = 16 \int \frac{\frac{2}{3} \cos \theta \, d\theta}{\left( \frac{2}{3} \sin \theta \right)^2 \sqrt{4 - 9 \left( \frac{2}{3} \sin \theta \right)^2}} = 24 \int \frac{\cos \theta \, d\theta}{2 \sin^2 \theta \cos \theta} = 12 \int \frac{1}{\sin^2 \theta} \, d\theta = -12 \cot \theta + C
\]

\[
\cot \theta = \frac{\cos \theta}{\sin \theta} = \frac{1 - \frac{9}{4}x^2}{\frac{3}{2}x} = \frac{\frac{1}{2} \sqrt{4 - 9x^2}}{3x} = \frac{\sqrt{4 - 9x^2}}{3x}
\]

\[
\int \frac{16}{x^2 \sqrt{4 - 9x^2}} \, dx = -\frac{4 \sqrt{4 - 9x^2}}{x} + C
\]
Integration by Parts:
\[ \int_{a}^{b} u \, dv = uv \bigg|_{a}^{b} - \int_{a}^{b} v \, du \]

Partial Fractions: If integrating \( \int \frac{P(x)}{Q(x)} \, dx \) where the degree of \( P(x) \) is smaller than the degree of \( Q(x) \). Factor denominator as completely as possible and find the partial fraction decomposition of the rational expression. Integrate the partial fraction decomposition (P.F.D.). For each factor in the denominator we get term(s) in the decomposition according to the following table.

<table>
<thead>
<tr>
<th>Factor in ( Q(x) )</th>
<th>Term in P.F.D</th>
<th>Factor in ( Q(x) )</th>
<th>Term in P.F.D</th>
</tr>
</thead>
<tbody>
<tr>
<td>( ax + b )</td>
<td>( \frac{A}{ax + b} )</td>
<td>( (ax + b)^{k} )</td>
<td>( \frac{A_{1}}{(ax + b)^{2}} + \cdots + \frac{A_{k}}{(ax + b)^{k}} )</td>
</tr>
<tr>
<td>( ax^{2} + bx + c )</td>
<td>( \frac{Ax + B}{ax^{2} + bx + c} )</td>
<td>( (ax^{2} + bx + c)^{k} )</td>
<td>( \frac{A_{1}x + B_{1}}{(ax^{2} + bx + c)^{2}} + \cdots + \frac{A_{k}x + B_{k}}{(ax^{2} + bx + c)^{k}} )</td>
</tr>
</tbody>
</table>

Ex. \( \int \frac{7x^{2}+13x}{(x-1)(x^{2}+4)} \, dx \)
\[ \int \frac{7x^{2}+13x}{(x-1)(x^{2}+4)} \, dx = \int \frac{4}{x-1} + \frac{3x-16}{x^{2}+4} \, dx \]
\[ = 4 \ln |x-1| + \frac{3}{2} \ln (x^{2}+4) + 8 \tan^{-1} \left( \frac{x}{2} \right) \]
Here is partial fraction form and recombined.

An alternate method that sometimes works to find constants. Start with setting numerators equal in previous example: \( 7x^{2} + 13x = A(x^{2} + 4) + (Bx + C)(x-1) \). Chose nice values of \( x \) and plug in.
For example if \( x = 1 \) we get \( 20 = 5A \) which gives \( A = 4 \). This won’t always work easily.

Even function: \( f(-x) = f(x) \)
\[ \frac{a}{-a} \int_{a}^{b} f(x) \, dx = \frac{a}{0} \int_{a}^{b} f(x) \, dx \]
Even function: \( f(-x) = -f(x) \)
\[ \frac{a}{-a} \int_{a}^{b} f(x) \, dx = 0 \]
Odd function: \( f(-x) = -f(x) \)
\[ \frac{a}{-a} \int_{a}^{b} f(x) \, dx = 0 \]
Improper Integral is an integral with one or more infinite limits and/or discontinuous integrands. Integral is called **convergent** if the limit exists and has a finite value or **divergent** if the limit doesn’t exist or has infinite value.

**Infinite Limit**

1. \[ \int_{a}^{\infty} f(x) \, dx = \lim_{b \to \infty} \int_{a}^{b} f(x) \, dx \]
2. \[ \int_{-\infty}^{b} f(x) \, dx = \lim_{a \to -\infty} \int_{a}^{b} f(x) \, dx \]
3. \[ \int_{-\infty}^{\infty} f(x) \, dx = \lim_{a \to -\infty} \int_{a}^{c} f(x) \, dx + \lim_{b \to \infty} \int_{c}^{b} f(x) \, dx \quad \text{providing BOTH integrals are convergent} \]

**Discontinuous Integrand**

1. **Discont. at** \( a \): \[ \int_{a}^{b} f(x) \, dx = \lim_{c \to a^{+}} \int_{a}^{c} f(x) \, dx + \lim_{c \to b^{-}} \int_{c}^{b} f(x) \, dx \]
2. **Discont. at** \( b \): \[ \int_{a}^{b} f(x) \, dx = \lim_{c \to b^{-}} \int_{a}^{c} f(x) \, dx + \lim_{c \to b^{+}} \int_{c}^{b} f(x) \, dx \]
3. **Discont. at** \( a < c < b \): \[ \int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx \]

**Useful Fact:** \( p - \) series integrals

- if \( a > 0 \), then \( \int_{a}^{\infty} \frac{1}{x^p} \, dx \) is convergent if \( p > 1 \) and divergent if \( p \leq 1 \).
- if \( a = 1 \) and \( p > 1 \), then \( \int_{1}^{\infty} \frac{1}{x^p} \, dx \) **converges** to \( \frac{1}{p-1} \).

**Comparison Test for Improper Integrals:** If \( 0 \leq f(x) \leq g(x) \) on \([a, \infty)\) then

1. If \( \int_{a}^{\infty} g(x) \, dx \) conv., then \( \int_{a}^{\infty} f(x) \, dx \) conv.
2. If \( \int_{a}^{\infty} f(x) \, dx \) divg., then \( \int_{a}^{\infty} g(x) \, dx \) divg.

**Mean Value Theorem for Integrals:**

If \( f(x) \) is continuous on \([a, b]\), then there exists a number \( c \) in \((a, b)\) such that

\[ f(c) = \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \]

**Restatement:** Under the given conditions, there is a point in the open interval where the value of the function is equal to the average value of the function over the interval.

**Geometric Interpretation:** Under the given conditions, there is a point in the open interval where the value of the function corresponds to the height of a rectangle, with base \((b - a)\), whose area is the same as the area under the curve between the two endpoints.
Applications of Integrals

**NET area:** \( \int_a^b |f(x)|\,dx = |A_1| + |A_2| \)

**NET accumulation:** \( \int_a^b f(x)\,dx = |A_1| - |A_2| \)

**Area between Curves:** The general formulas for the two main cases are:

\[
A = \int_a^b dA = \int_a^b [f(x) - g(x)]\,dx
\]
\[
A = \int_c^d dA = \int_c^d [f(y) - g(y)]\,dy
\]

If the curves intersect then the area of each portion must be found individually.

**Volumes of Revolution by slicing:** The two main formulas are \( V = \int A(x)\,dx \) and \( V = \int A(y)\,dy \)

Here is some general information about each method of computing and some examples.

**Disk:**

\[
V = \int_a^b A(x)\,dx = \pi \int_a^b [y(x)]^2\,dx
\]
\[
V = \int_c^d A(y)\,dy = \pi \int_c^d [x(y)]^2\,dy
\]

**Washers:**

\[
A = \pi \left\{ \left( \text{outer} \right)^2 - \left( \text{inner} \right)^2 \right\}
\]
\[ V = \int_a^b A(x)dx = \pi \int_a^b (y_1(x)^2 - y_2(x)^2)dx \quad \text{and} \quad V = \int_c^d A(y)dy = \pi \int_c^d (x_1(y)^2 - x_2(y)^2)dy \]

### Cylinders:

\[ V = \int_a^b A dy = 2\pi \int_a^b x[y_1(x) - y_2(x)]dx \]

**Differential Equations**

A differential equation is an equation involving derivatives of an unknown function and possibly the function itself as well as the independent variable.

The **order** of a differential equation is the highest order of the derivatives of the unknown function appearing in the equation

1. **Direct Integration** (the simplest cases)

   \[ y' = \sin(x) \Rightarrow y = -\cos(x) + C \]
   \[ y'' = 6x + e^x \Rightarrow y' = 3x^2 + e^x + C_1 \Rightarrow y = x^3 + e^x + C_2x + C_3 \]

2. **Separable Differential Equations**

   A separable differential equation can be expressed as the product of a function of \( x \) and a function of \( y \).

   \[ \frac{dy}{dx} = g(x) \cdot h(y) \quad \Rightarrow \quad \int \frac{dy}{h(y)} = \int g(x) \, dx + C \]

3. **Initial-value problem**

   is an ordinary differential equation together with a specified value of the unknown function at a given point in the domain of the solution

   If \( \frac{dy}{dx} = y \tan x \quad \text{and} \quad y = 3 \quad \text{when} \quad x = 0 \). What is \( y \) when \( x = \frac{\pi}{3} \)? (separable differential eq.)

   \[ \int \frac{dy}{y} = \int \frac{\sin x}{\cos x} \, dx + C \quad \Rightarrow \quad \ln|y| = - \int \frac{du}{u} + C \quad \Rightarrow \quad \ln|y| = - \ln|\cos x| + C \]

   \[ \ln |y \cos x| = \ln C \quad \Rightarrow \quad y \cos x = C \quad \& \quad (0, 3) \quad \Rightarrow \quad C = 3 \quad \Rightarrow \quad y \cos x = 3 \]

   when \( x = \frac{\pi}{3} \quad y = 6 \)
4. Exponential Growth and decay

rate of change is proportional to the amount present

\[
\frac{dP}{dt} = kP \quad \Rightarrow \quad P = P_0 e^{kt}
\]

Bacteria in a culture increased from 400 to 1600 in three hours. Assuming that the rate of increase is directly proportional to the population

a) Find an appropriate equation to model the population.
b) Find the number of bacteria at the end of six hours.

\[
\frac{dP}{dt} = kP \quad \Rightarrow \quad P = P_0 e^{kt}
\]

@ \( t = 0 \) \( P = P_0 = 400 \quad \Rightarrow \quad P = 400 e^{kt} \)

@ \( t = 3 \) \( P = 1600 \quad \Rightarrow \quad e^{3k} = 4 \quad \Rightarrow \quad e^k = 4^{\frac{1}{3}} \)

\( a) \quad P = 400 \left( 4^{\frac{t}{3a}} \right) \)

\( b) \quad @ \ t = 6 \quad P = 400 \left( 4^2 \right) = 6400 \)

If we were to solve for \( k \) (\( k = \frac{1}{3} \ln 4 = 0.462 \)) rather than \( e^k \) we might have the problem with rounding!!

If you decide to find \( k \), please keep all digits without rounding them. Round at the end. !!!!!!!!

Continuously Compounded Interest

interest is proportional to the amount present

\[
P' = kP \quad \Rightarrow \quad P = P_0 e^{kt}
\]

\( k \) is Interest rate

Radioactive Decay

amount of a radioactive element left after time \( t \) is proportional to amount present:

\[
-\frac{dN}{dt} = kt \quad \Rightarrow \quad N(t) = N_0 e^{-kt}
\]

In problems: The half-life is the time required for half the material to decay. \( k \) is not a half-life

Newton’s Law of Cooling

The rate of cooling is proportional to the difference in temperature between the liquid and surrounding.

\[
\frac{dT}{dt} = -k[T - T_s]
\]

\[
\left( \frac{dT}{dt} = \frac{d(T - T_s)}{dt} \right) \quad \Rightarrow \quad T - T_s = [T_0 - T_s]e^{-kt}
\]

5. Logistic Growth

Natural maximum exists, such that the growth cannot occur beyond it. There is a maximum population, or carrying capacity, \( M \).

Growth rate is proportional to the amount present \( (P) \), but also how far the current value is from the carrying capacity \( (M-P) \).

\[
\frac{dP}{dt} = kP(M - P) \quad \Rightarrow \quad P = \frac{M}{1 + Ce^{-(Mk)t}}
\]

In problems you get, you should push any equation toward the form \( P' = kP(M-P) \) to recognize \( M \) usually
Numerical Methods for Solving Differential Equations

**Euler’s Method:** \[ \frac{dy}{dx} = f(x, y) \quad y(x_0) = y_0 \]

<table>
<thead>
<tr>
<th>(x_n)</th>
<th>(f_n = f(x_n, y_n))</th>
<th>(y_{n+1} = y_n + f_n \Delta x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_0)</td>
<td>(\ldots)</td>
<td>(y_0)</td>
</tr>
<tr>
<td>(x_0 + \Delta x)</td>
<td>(f(x_0, y_0))</td>
<td>(y_1 = y_0 + f(x_0, y_0)\Delta x)</td>
</tr>
<tr>
<td>(x_1 + \Delta x)</td>
<td>(f(x_1, y_1))</td>
<td>(y_2 = y_1 + f(x_1, y_1)\Delta x)</td>
</tr>
<tr>
<td>(x_2 + \Delta x)</td>
<td>(f(x_2, y_2))</td>
<td>(y_3 = y_2 + f(x_2, y_2)\Delta x)</td>
</tr>
<tr>
<td>(x_{n+1})</td>
<td>(f(x_{n+1}, y_{n+1}))</td>
<td>(y_{n+1} = y_n + f(x_n, y_n)\Delta x)</td>
</tr>
</tbody>
</table>

Euler’s method follows pieces of approximate tangent lines. These values will be a bit low if \(f\) is concave up \((y'' = f' > 0)\) or a bit too large if \(f\) is concave down \((y'' = f' < 0)\)

**Slope Fields** show lots of little pieces (line segments) of the tangent lines for many possible curves with various possible initial conditions

---

**Parametric Equations and Curves**

parametric equations: \(x = f(t)\) \quad \(y = g(t)\)

Parametric curves have a direction of motion given by increasing \(t\).

\[
\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \frac{dy}{dt} \frac{dx}{dt} = \frac{dy}{dx} \quad \frac{d^2y}{dx^2} = \frac{d}{dx} \left[ \frac{dy}{dx} \right] = \frac{d}{dt} \left[ \frac{dy}{dx} \right] \frac{dt}{dx} = \frac{d}{dt} \left[ \frac{d}{dx} \right] \frac{dy}{dx} = \frac{d}{dx} \left[ \frac{dy}{dx} \right]
\]

**Polar Equations and Curves** \(r = r(\theta)\)

\[
x = r \cos \theta \quad y = r \sin \theta
\]

\[
r = \sqrt{x^2 + y^2} \quad \theta = \arctan \frac{y}{x}
\]
**ALL TOGETHER:**

**Derivatives:**

<table>
<thead>
<tr>
<th>Cartesian</th>
<th>Parametric</th>
<th>Polar</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{dy}{dx} )</td>
<td>( \frac{dy}{dt} \frac{dt}{dx} = \frac{dy}{dx} ) ( \frac{1}{dt} \frac{dx}{dt} = \frac{dy}{dx} \frac{dt}{dt} )</td>
<td>( x = r\cos\theta \Rightarrow \frac{dx}{d\theta} = \frac{dr}{d\theta} \cos \theta - r \sin \theta )</td>
</tr>
<tr>
<td>( \frac{d^2y}{dx^2} )</td>
<td>( \frac{d}{dx} \left[ \frac{dy}{dx} \right] = \frac{d}{dt} \left[ \frac{dy}{dx} \right] \frac{dt}{dx} )</td>
<td>( y = r\sin\theta \Rightarrow \frac{dy}{d\theta} = \frac{dr}{d\theta} \sin \theta + r\cos \theta )</td>
</tr>
</tbody>
</table>

\[
\frac{dy}{d\theta} = \frac{\frac{dy}{dx}}{\frac{dx}{d\theta}} = \frac{dr}{d\theta} \left(\frac{\sin \theta + r\cos \theta}{\cos \theta - r \sin \theta}\right)
\]

**Tangent line:** \( y = m(x - x_0) + y_0 \)  
**Normal line:** \( y = -\frac{1}{m}(x - x_0) + y_0 \)  

- **Cartesian form** at point \((x_0, y_0)\) of the curve \( y = y(x) \)  
- **Parametric form** at point \( t = t_0 \) of the curve \( x = x(t), y = y(t) \) \( \{x_0 = x(t_0), y_0 = y(t_0)\} \)  
- **Polar form** at point \( \theta = \theta_0 \) of the curve \( r = r(\theta) \) \( \{r_0 = r(\theta_0), x_0 = r_0 \cos \theta_0, y_0 = r_0 \sin \theta_0\} \)

**Concavity at point** \((x_1, y_1)\) or \( t_1 \) or \( \theta_1 \):
Find the second derivative \( \frac{d^2y}{dx^2} \) at that point.

If \( \frac{d^2y}{dx^2} < 0 \rightarrow \text{curve is concave down} \)  
If \( \frac{d^2y}{dx^2} > 0 \rightarrow \text{curve is concave up} \)

**Horizontal tangent line:** Horizontal tangent will occur where the derivative is zero: \( \frac{dy}{dx} = 0 \)

- **Cartesian plane:** \( y = y(x) \) \( \frac{dy}{dx} = 0 \Rightarrow x_0 \) (max or min) \( \Rightarrow y_0 = y(x_0) \)  
  \[\text{eq: } y = y_0\]

- **Parametric form:** \( x = x(t), y = y(t) \) \( \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dt} = 0 \Rightarrow t_0 \left(\check{\frac{dx}{dt}}\right|_{t_0} \neq 0 \right) \Rightarrow y_0 = y(t_0) \)  
  \[\text{eq: } y = y_0\]

- **Polar curve:** \( r = r(\theta) \) \( \frac{dy}{d\theta} = 0 \Rightarrow \theta_0 \left(\check{\frac{dx}{dt}}\right|_{\theta_0} \neq 0 \right) \Rightarrow r_0 = r(\theta_0) \Rightarrow y_0 = r_0 \sin \theta_0 \)  
  \[\text{eq: } y = y_0\]
**Vertical tangent line**: Vertical tangents will occur where the derivative is not defined:  
\[ \frac{dy}{dx} = \infty \]

**Cartesian plane**: \( y = y(x) \)  
\[ \frac{dy}{dx} = \infty \implies x_0 \]

**Parametric form**: \( x = x(t), \ y = y(t) \)  
\[ \frac{dy}{dx} = \infty \implies \frac{dx}{dt} = 0 \implies t_0 \left( \text{check } \frac{dy}{dt} \bigg|_{t_0} \neq 0 \right) \implies x_0 = x(t_0) \]

**Polar curve**: \( r = r(\theta) \)  
\[ \frac{dy}{dx} = \infty \implies \frac{dx}{d\theta} = 0 \implies \theta_0 \left( \text{check } \frac{dy}{d\theta} \bigg|_{\theta_0} \neq 0 \right) \implies r_0 = r(\theta_0) \implies x_0 = r_0 \cos \theta_0 \]

**Arc length**

- **Cartesian plane**: \( y = y(x) \)  
  \[ S = \int_a^b ds = \int_a^b \sqrt{(dx)^2 + (dy)^2} = \int_{t_1}^{t_2} \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \]

- **Parametric form**: \( x = x(t), \ y = y(t) \)  
  \[ S = \int_a^b ds = \int_a^b \sqrt{(dx)^2 + (dy)^2} = \int_{t_1}^{t_2} \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \, dt \]

- **Polar curve**: \( r = r(\theta) \)  
  \[ S = \int_{\theta_1}^{\theta_2} ds = \int_{\theta_1}^{\theta_2} r(\theta) \sqrt{r^2 + \left( \frac{dr}{d\theta} \right)^2} \, d\theta \]

\( t_1 \) and \( t_2 \) are times corresponding to positions \( a \) and \( b \)
\( \theta_1 \) and \( \theta_2 \) are angles corresponding to positions \( a \) and \( b \)

**Area**

- **Area enclosed by the curve and x axis**: \( y = y(x) \)  
  \[ A = \int_a^b \frac{1}{2} \, dy = \int_a^b \frac{1}{2} \, [y(x)] \, dx \]

- **Parametric form**: \( x = x(t), \ y = y(t) \)  
  \[ A = \int_a^b \frac{1}{2} \, \left[ y(x(t)) \right] \, dx = \int_{t_1}^{t_2} \frac{1}{2} \, y(t) \, \frac{dx}{dt} \, dt \]

\( t_1 \) and \( t_2 \) are times corresponding to positions \( a \) and \( b \)

- **Area enclosed by a polar curve \( r = r(\theta) \)**  
  \[ A = \int_{\theta_1}^{\theta_2} \frac{1}{2} \, r^2 \, d\theta \]

\( \theta_1 \) and \( \theta_2 \) are angles corresponding to positions \( a \) and \( b \)
Particle Motion

1. Suppose a particle moves along a smooth curve in the plane so that its position at any time \( t \) is \( \langle x(t), y(t) \rangle \), where \( x \) and \( y \) are differentiable functions of \( t \).

- \( \vec{s} = \langle x(t), y(t) \rangle \) is the particle's position vector at time \( t \)
- \( \langle x(t), y(t) \rangle = (x_0, y_0) + \vec{v}t \) is the particle's position moving with \( \vec{v} \) from starting position \( (x_0, y_0) \)
- \( \vec{v} = \left( \frac{dx}{dt}, \frac{dy}{dt} \right) \equiv \langle x'(t), y'(t) \rangle \) is the velocity vector at time \( t \)
- \( \vec{a} = \langle x''(t), y''(t) \rangle \) is the acceleration vector at time \( t \)

- \( |\vec{v}| \equiv \|\vec{v}\| = \sqrt{(x'(t))^2 + (y'(t))^2} \) is the speed of a particle [magnitude/length/norm of the velocity vector].

- \( |\vec{a}| \equiv \|\vec{a}\| = \sqrt{(x''(t))^2 + (y''(t))^2} \) is the magnitude/length/norm of acceleration.

- \( \vec{v} \left/ |\vec{v}| \right. \) is the direction vector representing the particle's direction of motion

2. Suppose a particle moves along a path in the plane so that its velocity at any time \( t \) is \( \vec{v} = \langle x'(t), y'(t) \rangle \)

- \( \vec{s} = \int_{t_1}^{t_2} \vec{v}(t) \, dt \) is displacement covered from \( t_1 \) to \( t_2 \)

- \( \int_{t_1}^{t_2} |\vec{v}(t)| \, dt = \int_{t_1}^{t_2} \sqrt{(v_x)^2 + (v_y)^2} \, dt = \int_{t_1}^{t_2} \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \, dt \) is distance covered from \( t_1 \) to \( t_2 \)

- \( \langle x, y \rangle = \langle x_0, y_0 \rangle + \int_{t_1}^{t_2} \vec{v}(t) \, dt \) is new position

**Parametric Motion**: Given two parametric functions \( x(t) \) and \( y(t) \), or a vector function \( \langle x(t), y(t) \rangle \), that describe the motion of an object. At any point \( t \) for which:

- \( y'(t) = 0 \) and \( x'(t) \neq 0 \), the motion will be parallel to the \( x \)-axis and the curve will have a horizontal tangent.
- \( y'(t) \neq 0 \) and \( x'(t) = 0 \), the motion will be parallel to the \( y \)-axis and the curve will have a vertical tangent.
- \( y'(t) = 0 \) and \( x'(t) = 0 \), the motion is stopped. If neither \( x'(t) \) nor \( y'(t) \) change sign at the point, the curve will have a tangent line. Otherwise there will be no tangent line.

**Distance versus Displacement**:

Displacement (or change in position) = \( \int_{a}^{b} v(t) \, dt \) \hspace{1cm} Total Distance Traveled = \( \int_{a}^{b} |v(t)| \, dt \).
SEQUENCES AND SERIES

SEQUENCES

DEF 1 A sequence \( \{a_n\} \) has the limit \( L \)
\[
\lim_{n \to \infty} a_n = L \quad \text{or} \quad a_n \to L \quad \text{as} \quad n \to \infty
\]
if we can make the terms \( a_n \) as close to \( L \) as we like by taking \( n \) sufficiently large.

If \( \lim_{n \to \infty} a_n \) exists, we say the sequence is convergent.

Otherwise, we say the sequence is divergent.

THEOREM If \( \lim_{x \to \infty} |a_n| = 0 \), then \( \lim_{n \to \infty} a_n = 0 \)

This thm is true only if \( L = 0 \)

SERIES

DEF

If the sequence \( \{S_n\} \) is convergent and \( \lim_{n \to \infty} S_n = S \) exists as a real number,

then the series \( \{a_n\} \) is called convergent and we write

\[
\sum_{n=1}^{\infty} a_n = S
\]

- \( S \) is the sum of the series.

If \( \lim_{n \to \infty} S_n \) is nonexistent, then the series \( \sum_{n=1}^{\infty} a_n \) diverges and has no sum.

Summary of Tests for Infinite Series Convergence \( \left( \sum_{n=1}^{\infty} a_n \quad \text{or} \quad \sum_{n=0}^{\infty} a_n \right) \)

**n**th – term test

If \( \lim_{n \to \infty} a_n \neq 0 \), then the series is divergent.

If \( \lim_{n \to \infty} a_n = 0 \), then the series may converge or diverge, so you need to use different test.

**Geometric Series Test**

If the series has the form \( \sum_{n=1}^{\infty} a r^{n-1} \) or \( \sum_{n=0}^{\infty} a r^n \),

then the series converges if \( |r| < 1 \) and diverges otherwise

If the series converges, then it converges to \( \frac{a}{1-r} \)

\[
S = \sum_{n=0}^{\infty} a r^n = \frac{a}{1-r}
\]
p-series test (proof by integral test)

If the series has the form \( \sum_{n=1}^{\infty} \frac{1}{n^p} \), then the series converges if \( p > 1 \) and diverges otherwise.

When \( p = 1 \), the series is the divergent Harmonic series.

Integral Test

If \( a_n = f(n) \) and \( f(x) \) is continuous, positive and decreasing function on the interval \([1, \infty)\)

then if improper integral \( \int_{1}^{\infty} f(x) \, dx \) converges, \( \sum_{n=1}^{\infty} a_n \) converges.

then if improper integral \( \int_{1}^{\infty} f(x) \, dx \) diverges, \( \sum_{n=1}^{\infty} a_n \) diverges.

If convergent, upper bound is: \( \sum_{n=1}^{\infty} a_n \leq a_1 + \int_{1}^{\infty} f(x) \, dx \)

\( S = \sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + a_{n+1} + a_{n+2} + a_{n+3} + \cdots = S_n + R_n \)

\( S_n \) is Estimation of the sum \( S \), and \( R_n \) is the Remainder

\[
\int_{n+1}^{\infty} f(x) \, dx \leq R_n \leq \int_{n}^{\infty} f(x) \, dx \quad \Rightarrow \quad S_n + \int_{n+1}^{\infty} f(x) \, dx \leq S \leq S_n + \int_{n}^{\infty} f(x) \, dx
\]

Direct Comparison Test (DCT)

- If \( a_n \leq b_n \) and \( \sum_{n=1}^{\infty} b_n \) converges, then \( \sum_{n=1}^{\infty} a_n \) converges.

- If \( a_n \geq b_n \) and \( \sum_{n=1}^{\infty} b_n \) diverges, then \( \sum_{n=1}^{\infty} a_n \) diverges.

Ex: compare \( \frac{n^2}{2^n} \) to \( \frac{1}{2^n} \), \( \frac{1}{n^3 + 1} \) to \( \frac{1}{n^3} \), \( \frac{n^2}{(n^2 + 3)^2} \) to \( \frac{n}{(n^2 + 3)^2} \)

Limit Comparison Test (LCT)

(may be used instead of DCT most of the time)

If \( a_n, b_n > 0 \) and \( \lim_{n \to \infty} \frac{|a_n|}{|b_n|} = \text{finite number} \)

then either both \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) converge or diverge.

Use this test when you cannot compare term by term because the rule of sequence is “too UGLY” but you can still find a known series to compare with it.

Ex: compare \( \frac{3n^2 + 2n - 1}{4n^5 - 6n + 7} \) to \( \frac{1}{n^3} \) (you can disregard the leading coefficient and all nonleading terms, looking only at the condensed degree of the leading terms: \( \frac{n^2}{n^3} = \frac{1}{n} \).
**Ratio Test**

If \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L \) (real number or \( \infty \)) , then

(i) \( \sum a_n \) converges absolutely (and hence converges) if \( L < 1 \)

(ii) \( \sum a_n \) diverges if \( L > 1 \) (or \( \infty \))

(iii) the test is inconclusive if \( L = 1 \) (use another test)

Use this test for series whose terms converge rapidly, for instance those involving exponentials and/or factorials!!

**Root Test**

If \( \lim_{n \to \infty} \sqrt[n]{|a_n|} = L \) (real number or \( \infty \)) , then

(i) \( \sum a_n \) converges absolutely (and hence converges) if \( L < 1 \)

(ii) \( \sum a_n \) diverges if \( L > 1 \) (or \( \infty \))

(iii) the test is inconclusive if \( L = 1 \) (use another test)

Use this test for series involving \( n^{th} \) powers. Ex) \( \sum \frac{e^{2n}}{n^n} \)

**Alternating series test**

If the series has the form \( \sum a_n = \sum (-1)^{n-1} b_n \quad b_n > 0 \), then the series converges if

i) \( 0 \leq b_{n+1} \leq b_n \quad \text{for all} \ n \in \mathbb{Z}^+ \quad \text{(decreasing terms)} \)

ii) \( \lim_{n \to \infty} b_n = 0 \)

If either of these conditions fails, the test fails, and you need use a different test.

If convergent, a partial sum \( S_n \) is used as an **approximation** to the total sum \( S \).

\( R_n \) is the Remainder: \( |R_n| = |S - S_n| \leq b_{n+1} \), and the sum \( S \) lies: \( S_n - b_{n+1} \leq S \leq S_n + b_{n+1} \)

**ABSOLUTE AND CONDITIONAL CONVERGENCE** (for LOCO series \( +,-,+,-,+,+,+,-,-,- \))

If \( \sum_{n=1}^{\infty} |a_n| \) converges, then \( \sum_{n=1}^{\infty} a_n \) is **Absolutely Convergent**.

If \( \sum_{n=1}^{\infty} a_n \) is absolutely convergent then it is (just) convergent. The converse of this statement is false.

If \( \sum_{n=1}^{\infty} |a_n| \) diverges but \( \sum_{n=1}^{\infty} a_n \) converges, then \( \sum_{n=1}^{\infty} a_n \) is **Conditionally Convergent**

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \] is conditionally convergent. Harmonic series is divergent, alternating harmonic series is convergent.
Remember, if you are asked to find the ACTUAL sum of an infinite series, it must either be a Geometric series \( S = \frac{a}{1-r} \) or a Telescoping Series (requires expanding and canceling terms).

The only other tests that allows us to approximate the infinite sum are the Integral test and the Alternate Series Test. We can find the \( n \)th partial sum \( S_n \) for any series.

### Convergence of Power Series

**For a power series centered at** \( a \), **precisely one of the following is true:**

1. **The series converges only at** \( x = a \) (\( \Sigma c_n \) \( (x - a)^n = c_0 \); **radius of convergence** \( R = 0 \)

2. **There exist a real number** \( R > 0 \) **such that**
   - the series converges absolutely for \( |x - a| < R \)
   - and diverges for \( |x - a| > R \);
   - **\( R \) is radius of convergence**

3. **The series converges absolutely for all** \( x \); **radius of convergence** \( R = \infty \)

   The set of values of \( x \) for which series converges is the **interval of convergence**

**Logistics:**

1. From Ratio Test you can find radius of convergence and **open** interval of convergence.

2. To test two points you go back to the power series and plug in the endpoints and check convergence

   **Find the interval of convergence of** \( \sum_{n=0}^{\infty} \frac{(-1)^n(x + 1)^n}{2^n} \)

   \[
   \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(x + 1)^{n+1}}{2^{n+1}} \cdot \frac{2^n}{(x + 1)^n} \right| = \left| \frac{x + 1}{2} \right|
   \]

   by ratio test, the series converges if \( \left| \frac{x + 1}{2} \right| < 1 \) or \( |x + 1| < 2 \) \( \rightarrow R = 2 \) \( \rightarrow -3 < x < 1 \)

   - \( x = -3: \quad \sum_{n=0}^{\infty} \frac{(-1)^n(2)^n}{2^n} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2^n} = \sum_{n=0}^{\infty} 1 \) series diverges
   - \( x = 1: \quad \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{2^n} = \sum_{n=0}^{\infty} (-1)^n \) series diverges

   **interval of convergence:** \( x \in (-3, 1) \)

### TAYLOR SERIES

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n \approx \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x - a)^k
\]

error is remainder \( R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!} (x - a)^{n+1} \)

\( z \) is the value between \( a \) and \( x \), that yields maximum \( R_n \)