1. AREA BETWEEN the CURVES

\[ dA = \left\{ \left( \text{outer function} \right) - \left( \text{inner function} \right) \right\} \, dx \]

\[ A = \int_a^b dA = \int_a^b \left[ y_1(x) - y_2(x) \right] dx \]

\[ A = \int_c^d dA = \int_c^d \left[ x_1(y) - x_2(y) \right] dy \]

**EX:** Determine the area of the region bounded by \( y = 2x^2 + 10 \) and \( y = 4x + 16 \) between \( x = -2 \) and \( x = 5 \)

\[ A = \int_{-2}^5 dA = \int_{-2}^5 \left[ \left( \text{outer function} \right) - \left( \text{inner function} \right) \right] \, dx \]

\[ A = \int_{-2}^5 \left[ 2x^2 + 10 - (4x + 16) \right] \, dx + \int_4^5 \left[ 4x + 16 - (2x^2 + 10) \right] \, dx \]

\[ = \frac{14}{3} + \frac{64}{3} + \frac{64}{3} = \frac{142}{3} \]
EX: Determine the area of the region enclosed by \( y = \sin x \) and \( y = \cos x \) and the \( y \)-axis for \( 0 \leq x \leq \frac{\pi}{2} \)

\[
A = \int_0^{\pi/2} [\cos x - \sin x] \, dx + \int_0^{\pi/4} [\sin x - \cos x] \, dx \\
= (\sin x + \cos x)\bigg|_0^{\pi/2} + (\cos x - \sin x)\bigg|_0^{\pi/4} \\
= \sqrt{2} - 1 + (\sqrt{2} - 1) = 2\sqrt{2} - 2 = 0.828427
\]

EX: Determine the area of the enclosed area by \( x = \frac{1}{2} y^2 - 3 \) and \( y = x - 1 \)

Intersection: \((-1,-2)\) and \((5,4)\).

\[
A = \int_{-1}^{\frac{\sqrt{2}}{2}} \left( \sqrt{2x + 6} - \left( \sqrt{2x + 6} - (x-1) \right) \right) \, dx \\
= \int_{-1}^{\frac{\sqrt{2}}{2}} 2\sqrt{2x + 6} \, dx + \int_{-1}^{\frac{\sqrt{2}}{2}} \left( x - 1 \right) \, dx \\
= \frac{2}{3} \left[ y^3 \right]_{-1}^{\frac{\sqrt{2}}{2}} + \frac{1}{3} \left[ \frac{3}{2} y^2 + \frac{1}{2} y^2 + x \right]_{-1}^{\frac{\sqrt{2}}{2}} = 18
\]

THE SAME: Determine the area of the enclosed area by \( x = \frac{1}{2} y^2 - 3 \) and \( y = x - 1 \)

\[
A = \int_{-2}^{4} \left[ \left( y + 1 \right) - \left( \frac{1}{2} y^2 - 3 \right) \right] \, dy \\
= \left( \frac{1}{6} y^3 + \frac{1}{2} y^2 + 4y \right)\bigg|_{-2}^{4} = 18
\]

So, in this last example we’ve seen a case where we could use either method to find the area.
However, the second was definitely easier.
**EX:** Find area

Intersection points are:

\( y = -1 \)
\( y = 3 \)

\[
A = \int_{-1}^{3} [(-y^2 + 10) - (y - 2)^2] \, dy
= \int_{-1}^{3} [-2y^2 + 4y + 6] \, dy
= \left[ -\frac{2}{3}y^3 + 2y^2 + 6y \right]_{-1}^{3}
A = \frac{64}{3}
\]

**Volume of REVOLUTION**

- Find the Volume of revolution using the disk method
- Find the volume of revolution using the washer method
- Find the volume of revolution using the shell method
- Find the volume of a solid with known cross sections

Area is only one of the applications of integration. We can add up representative volumes in the same way we add up representative rectangles. When we are measuring volumes of revolution, we can slice representative disks or washers.

**DISK METHOD**

\[
V = \int_{a}^{b} dV \quad dV = \pi r^2 \, dx
\]

\[
dV = \pi [f(x)]^2 \, dx
\]

\[
dV = \pi [f(x) + k]^2 \, dx
\]
\[ V = \int_a^b dV \]

\[ dV = \pi \left[ f(x) - k \right]^2 dx \]

\[ dV = \pi \left[ k - f(x) \right]^2 dx \]

**WAHIER METHOD**

*A solid obtained by revolving a region around a line.*

\[ A = \pi \left( \text{outer radius}^2 - \text{inner radius}^2 \right) \]

\[ dV = \pi \left( y_2(x)^2 - y_1(x)^2 \right) dx \]

\[ dV = \pi \left( x_2(y)^2 - x_1(y)^2 \right) dy \]

**NOTE:** Cross-section is perpendicular to the axis of rotation.

\[ V = \int_a^b dV \quad dV = A \, dx \]

\[ V = \int_c^d dV \quad dV = A \, dy \]
Example:

Find the volume of the solid formed by revolving the region bounded by $y = \sqrt[4]{x}$ and $y = x^2$ over the interval $[0, 1]$ about the $x$-axis.

\[ V = \pi \int_0^1 \left( (\sqrt[4]{x})^2 - (x^2)^2 \right) dx \]

\[ V = \pi \left( \frac{x^2}{2} - \frac{x^5}{5} \right)_0^1 \]

\[ V = \frac{3}{10} \]

Example: rotate it around $x = \text{axis}$

\[ V = \pi \int_0^1 \left( (\sqrt{x-1})^2 - (x-1)^2 \right) dx \]

\[ = \pi \int_0^1 \left( x - 1 - (x-1)^2 \right) dx = \pi \left[ \frac{x^2}{2} - x - \frac{x}{3} \right]_0^1 \]

\[ = \pi \left( \frac{3}{2} - 1 - \frac{1}{3} \right) = \pi \left( \frac{3}{2} - 1 \right) = \frac{3\pi}{2} \]
Volumes by Cylindrical Shells

Summing up the volumes of all these infinitely thin shells, we get the total volume of the solid of revolution:

\[ A = 2\pi rh = 2\pi x h \]

\[ dV = A \, dx = 2\pi x h \, dx \]

\[ V = \int_a^b dV \]

\[ dV = A \, dx = 2\pi x \, y(x) \, dx \]

\[ V = 2\pi \int_a^b x \, y(x) \, dx \]

\[ dV = A \, dx = 2\pi x \, [y_1(x) - y_2(x)] \, dx \]

\[ V = 2\pi \int_a^b x [y_1(x) - y_2(x)] \, dx \]

Example: Find the volume of the solid of revolution formed by rotating the region bounded by the x-axis and the graph of \( y = \sqrt{x} \) from \( x = 0 \) to \( x = 1 \), about the y-axis.

\[ V = 2\pi \int_0^1 x \sqrt{x} \, dx \]

\[ V = 2\pi \left[ \frac{2}{5} x^{5/2} \right]_0^1 \]

\[ V = \frac{4\pi}{5} \]
Example: Find the volume of the solid of revolution formed by rotating the finite region bounded by the graphs of $y = \sqrt{x - 1}$ and $y = (x - 1)^2$ about the $y$-axis.

\[
V = 2\pi \int_{0}^{2} x(\sqrt{x - 1} - (x - 1)^2)dx
\]

\[
V = \frac{29\pi}{30}
\]

Time to Practice!!! Again

Example: Find the volume of the solid obtained by rotating the region bounded by $y = x - x^3$ and $y = 0$ about the line $x = 2$

\[
V = \int_{0}^{2} 2\pi (2 - x)(x - x^3) dx = 2\pi \int_{0}^{2} (x^3 - 3x^2 + 2x) dx = 2\pi \left[ \frac{x^4}{4} - x^3 + 2x \right]_{0}^{2} = \frac{\pi}{2}
\]

Example: rotate it around $x$ axis
The Volume for Solids with Known Cross Sections

Procedure: volume by slicing: sketch the solid and a typical cross section. Find a formula for the area, $A(x)$, of the cross section. Find limits of integration and integrate $A(x)$ to get volume.

Find the volume of a solid whose base is the circle $x^2 + y^2 = 4$ and where cross sections perpendicular to the x-axis are:

a) squares

$$x^2 + y^2 = 4 \quad y = \sqrt{4 - x^2}$$

Length of a side is: $2\sqrt{4 - x^2}$

$$dV = A \, dx \quad A = a^2$$

$$V = 4 \int_{-2}^{2} (4 - x^2) \, dx = \frac{128}{3}$$

b) Equilateral triangles

$$x^2 + y^2 = 4 \quad y = \sqrt{4 - x^2}$$

$$A = \frac{1}{2} a \sqrt{a^2 - \left(\frac{a}{2}\right)^2} = \frac{\sqrt{3}}{4} a^2 = \sqrt{3}(4 - x^2)$$

$$V = \int_{-2}^{2} \sqrt{3}(4 - x^2) \, dx = \frac{32}{\sqrt{3}} \approx 18.475$$

c) Semicircles

$$x^2 + y^2 = 4 \quad y = \sqrt{4 - x^2}$$

$$A = \frac{1}{2} \pi \left(\frac{a}{2}\right)^2 = \frac{1}{8} \pi \, a^2 = \pi \frac{4 - x^2}{2}$$

$$dV = A \, dx$$

$$V = \int_{-2}^{2} \pi \frac{4 - x^2}{2} \, dx = \frac{16\pi}{3} \approx 16.755$$
d) Isosceles right triangles

\[ x^2 + y^2 = 4 \quad y = \sqrt{4 - x^2} \]

\[ A = \frac{1}{2} a \left( \frac{a}{2} \right) = \frac{a^2}{4} = 4 - x^2 \]

\[ dV = A \, dx \]

\[ V = \int_{-2}^{2} (4 - x^2) \, dx = \frac{32}{3} \approx 10.667 \]

The base of the volume of a solid is the region bounded by the curve \( g(x) = 4x - x^2 \) and \( f(x) = x^2 \). Find the volumes of the solids whose cross sections perpendicular to the x-axis are the following:
PRACTICE:

1. Find the volume of the solid generated by revolving about the x-axis the region bounded by the graph of \( f(x) = \sqrt{x - 1} \) the x-axis, and the line \( x = 5 \). Draw a sketch. 1. ANS: 8\pi

2. Find the volume of the solid generated by revolving about the x-axis the region bounded by the graph of \( y = \sqrt{\cos x} \) where \( 0 \leq x \leq \frac{\pi}{2} \) the x-axis, and the y-axis. Draw a sketch. 2. ANS: \pi

3. Find the volume of the solid generated by revolving about the y-axis the region in the first quadrant bounded by the graph of \( y = x^2 \), the y-axis, and the line \( y = 6 \). Draw a sketch. 3. ANS: 18 \pi

4. Using a calculator, find the volume of the solid generated by revolving about the line \( y = 8 \) the region bounded by the graph of \( y = x^2 + 4 \), the line \( y = 8 \). Draw a sketch. ANS: 512/15 \pi

5. Using a calculator, find the volume of the solid generated by revolving about the line \( y = -3 \) the region bounded by the graph of \( y = e^x \), the y-axis, the lines \( x = \ln 2 \) and \( y = -3 \). Sketch. 5. ANS: 13.7383 \pi

6. Using the Washer, find the volume of the solid generated by revolving the region bounded by \( y = x^3 \) and \( y = x \) in the first quadrant about the x-axis. Draw a sketch. Method (just a fancy name – use sketch and common sense!!! instead of given boundaries, you have to find it as intersection of two curves and then use sketch to subtract one volume from another ) 6. ANS: 4\pi/21

7. Using the Washer Method and a calculator, find the volume of the solid generated by revolving the region bounded by \( y = x^3 \) and \( y = x \) about the line \( y = 2 \). Draw a sketch. 7. ANS: 17\pi/21

8. Using the Washer Method and a calculator, find the volume of the solid generated by revolving the region bounded by \( y = x^2 \) and \( x = y^2 \) about the y-axis. Draw a sketch. 8. ANS: 3\pi/10

AGAIN PRACTICE:

1. The base of a solid is the region enclosed by the ellipse \( \frac{x^2}{4} + \frac{y^2}{25} = 1 \). The cross sections are perpendicular to the x-axis and are isosceles right triangles whose hypotenuses are on the ellipse. Find the volume of the solid. 1. ANS: \( V = 200/3 \)

2. The base of a solid is the region enclosed by a triangle whose vertices are (0, 0), (4, 0) and (0, 2). The cross sections are semicircles perpendicular to the x-axis. Using a calculator, find the volume of the solid.
2. ANS: \( V = 2.094 \)

3. Find the volume of the solid whose base is the region bounded by the lines \( x + 4y = 4, \ x = 0, \) and \( y = 0 \), if the cross sections taken perpendicular to the \( x \)-axis are semicircles.

3. ANS: \( V = \frac{\pi}{6} \)

4. The base of a solid is the region in the first quadrant bounded by the \( y \)-axis, the graph of \( y = \arctan x \), the horizontal line \( y = 3 \), and the vertical line \( x = 1 \). For this solid, each cross section perpendicular to the \( x \)-axis is a square. What is the volume of the solid? 4. ANS: \( V = \int_0^1 (3 - \arctan(x))^2 \ dx = 6.61233 \)

5. A solid has its base is the region bounded by the lines \( x + 2y = 6, \ x = 0 \) and \( y = 0 \) and the cross sections taken perpendicular to \( x \)-axis are circles. Find the volume the solid.

5. ANS: \( \frac{9}{2} \pi \)

6. A solid has its base is the region bounded by the lines \( x + y = 4, \ x = 0 \) and \( y = 0 \) and the cross section is perpendicular to the \( x \)-axis are equilateral triangles. Find its volume. 6. ANS: \( V = \frac{16\sqrt{3}}{3} \)