Related Rates

Objectives: Identify a mathematical relationship between quantities that are each changing. Use one or more rates to determine another rate.

Process:
Step 1: Draw a diagram.

Step 2: Determine which quantities and rates are given, and which to be found.

Step 3: Identify the primary function to use. (Often this is a formula from geometry.)

Step 4: Differentiate with respect to the independent variable

Step 5: Write a related rates equation

Step 6: Substitute known quantities and solve for desired rate. [NOTE: Do not substitute known quantities before this last step!]

Type 1: Explicit Function of One Variable

Examples
1. Air is being blown into a sphere at the rate of 6 cubic inches per minute. How fast is the radius changing when the radius of the sphere is 2 inches?

\[ v = \frac{4}{3} \pi r^3 \]
\[ \frac{dv}{dt} = 4\pi r^2 \frac{dr}{dt} \]
\[ 6 = 4\pi (2)^2 \frac{dr}{dt} \]
\[ \frac{dr}{dt} = \frac{6}{16\pi} = \frac{1}{4\pi} \text{ in/min} \]

2. The edge of a cube is increasing at a rate of 2 inches per minute. At the instant the edge is 3 inches, how fast is the volume increasing?

\[ v = s^3 \]
\[ \frac{dv}{dt} = 3s^2 \frac{ds}{dt} \]
\[ 2 = 3(3)^2 \frac{ds}{dt} \]
\[ \frac{ds}{dt} = \frac{2}{27} \text{ in}^3/\text{min} \]
3. A point moves along the curve \( y = (x - 3)^2 \) such that its x-coordinate is increasing at 4 units per second.
   (a) At the moment \( x = 1 \), how fast is the y-coordinate changing? Interpret your answer based on the shape of the graph and the location of the point.

   What we know:
   \[
   \begin{align*}
   \frac{dx}{dt} &= 4 \text{ units/sec} \\
   y &= (x - 3)^2 \\
   \frac{dy}{dt} &= 2(x - 3) \frac{dx}{dt} \\
   \frac{dy}{dt} &= 2(1 - 3) \cdot 4 = -16 \text{ units/sec}
   \end{align*}
   \]

   (b) At the moment \( x = 1 \), how fast is the point’s distance from the origin changing?

   What we know:
   \[
   \begin{align*}
   \frac{dx}{dt} &= 4 \text{ units/sec} \\
   x &= 1 \\
   D &= \sqrt{(x - 0)^2 + ((x - 3)^2 - 0)^2} \\
   D &= \sqrt{x^2 + (x - 3)^4} \\
   D &= \sqrt{(x^2 + (x - 3)^4)^{1/2}} \\
   \frac{dD}{dt} &= \frac{1}{2} \left( x^2 + (x - 3)^4 \right)^{1/2} \cdot \left( 2x + 4(x - 3)^3 \right) \frac{dx}{dt} \\
   \frac{dD}{dt} &= \frac{1}{2} \left( 1 + (1 - 3)^4 \right)^{1/2} \cdot \left( 2(1) + 4(1 - 3)^3 \right) \cdot 4 \\
   \frac{dD}{dt} &= \frac{1}{2} \left( 17 \right)^{1/2} \cdot (- 30) \cdot 4 = -60(17)^{1/2} = -14.552 \text{ units/min}
   \end{align*}
   \]

Type 2 - Implicit Function of One Variable

4. A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/sec, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?

   What we know:
   \[
   \begin{align*}
   \frac{dx}{dt} &= 1 \text{ ft/sec} \\
   r &= 10 \text{ ft} \\
   x &= 6 \text{ ft} \\
   x^2 + y^2 &= r^2 \\
   2x \frac{dx}{dt} + 2y \frac{dy}{dt} &= 2r \frac{dr}{dt} \\
   (2 \cdot 6 \cdot 1) + \left( 2 \cdot 8 \cdot \frac{dy}{dt} \right) &= 0 \\
   16 \frac{dy}{dt} &= -12 \\
   \frac{dy}{dt} &= -\frac{12}{16} = -\frac{3}{4} \text{ ft/sec}
   \end{align*}
   \]
5. The altitude of a triangle is increasing at a rate of 1 cm/min while the area of the triangle is increasing at a rate of 2 cm²/min. At what rate is the base of the triangle changing when the altitude is 10 cm and the area is 100 cm²?

What we know:
\[
\frac{dh}{dt} = 1 \text{ cm/min} \\
\frac{dA}{dt} = 2 \text{ cm}^2/\text{min} \\
h = 10 \text{ cm} \\
A = 100 \text{ cm}^2
\]

\[
A = \frac{1}{2} bh \\
100 = \frac{1}{2} b \cdot 10 \\
b = 20 \text{ cm} \\
A = \frac{1}{2} bh \\
\frac{dA}{dt} = \frac{1}{2} \left( b \cdot \frac{dh}{dt} + h \cdot \frac{db}{dt} \right) \\
2 = \frac{1}{2} \left( 20 \cdot 1 + 10 \cdot \frac{db}{dt} \right) \\
\frac{db}{dt} = -1.6 \text{ cm/sec}
\]

6. A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of 2 m³/min, find the rate at which the water level is rising when the water is 3 m deep.

What we know:
\[
r_{\text{base}} = 2 \text{ m} \\
h_{\text{total}} = 4 \text{ m} \\
\frac{dv}{dt} = 2 \text{ m}^3/\text{min} \\
h = 3 \text{ m} \\
\frac{v}{12} = \frac{r}{h^4} \\
\frac{dv}{dt} = \frac{1}{4} \pi h^2 \frac{dh}{dt} \\
2 = \frac{1}{4} \pi (3)^2 \frac{dh}{dt} \\
\frac{dh}{dt} = \frac{8}{9\pi} \text{ m/min}
\]
Mixed Problem Set- Related Rates

1. A conical tank is being filled with water. The tank has height 4 ft and radius 3 ft. If water is being pumped in at a constant rate of 2 cubic inches per minute, find the rate at which the height of the cone changes when the height is 26 inches. Note the difference in units.

   What we know:

   \[
   \begin{align*}
   r_{\text{base}} &= 3 \text{ in} \\
   h_{\text{total}} &= 4 \text{ in} \\
   \frac{dv}{dt} &= 2 \text{ in}^3/\text{min} \\
   h &= 26 \text{ in}
   \end{align*}
   \]

   \[
   \begin{align*}
   r &= \frac{3}{4} h \\
   \frac{dv}{dt} &= \frac{9}{4} \pi h^2 \frac{dh}{dt} \\
   \frac{dv}{dt} &= 2 \pi r^2 h \\
   \frac{dh}{dt} &= \frac{2}{1521 \pi} \text{ in/ min}
   \end{align*}
   \]

2. A searchlight is positioned 10 meters from a sidewalk. A person is walking along the sidewalk at a constant speed of 2 meters per second. The searchlight rotates so that it shines on the person. Find the rate at which the searchlight rotates when the person is 25 meters from the searchlight.

   What we have:

   \[
   \begin{align*}
   y &= 10 \text{ m} \\
   \frac{dx}{dt} &= 2 \text{ m/sec} \\
   r &= 25 \text{ m}
   \end{align*}
   \]

   \[
   \begin{align*}
   x^2 + y^2 &= r^2 \\
   x^2 + (10)^2 &= (25)^2 \\
   x &= \sqrt{525} \\
   \tan \theta &= \frac{x}{y} \\
   \sec \theta &= \frac{25}{10} = \frac{5}{2} \\
   \frac{(5)^2}{2} \frac{d\theta}{dt} &= \frac{10 \cdot 2 - \sqrt{525} \cdot 0}{10^2} \\
   \frac{25}{4} \frac{d\theta}{dt} &= \frac{1}{5} \\
   \frac{d\theta}{dt} &= \frac{1}{25} \text{ rad/sec}
   \end{align*}
   \]

3. A person 5 feet tall is walking toward an 18-foot pole. A light is positioned at the top of the pole. Find the rate at which the length of the person’s shadow is changing when the person is 30 feet from the pole and walking at a constant speed of 6 feet per second.

   What we have:

   \[
   \begin{align*}
   \frac{18}{5} &= \frac{r + x}{r} \\
   \frac{18}{5} &= \frac{r + x}{x} \\
   \frac{18}{5} &= \frac{r + 30}{r}
   \end{align*}
   \]

   \[
   \begin{align*}
   18r &= 5r + 150 \\
   13r &= 150 \\
   r &= \frac{150}{13}
   \]

   \[
   \begin{align*}
   0 &= \frac{dx}{dt} - \frac{x}{r} \frac{dr}{dt} \\
   0 &= 0 + \frac{dx}{dt} \frac{dr}{dt} \\
   \frac{dx}{dt} &= -\frac{5}{18} \frac{dr}{dt}
   \end{align*}
   \]
4. The length of a rectangle increases by 3 feet per minute while the width decreases by 2 feet per minute. When the length is 15 feet and the width is 40 feet, what is the rate at which the following changes:

a. area: 
\[ A = LW \]
\[ \frac{dA}{dt} = L \frac{dW}{dt} + W \frac{dL}{dt} \]
\[ \frac{dA}{dt} = 15 \cdot (-2) + 40 \cdot 3 = 90 \text{ ft}^2 \text{ / min} \]

b. perimeter: 
\[ P = 2L + 2W \]
\[ \frac{dP}{dt} = 2 \frac{dL}{dt} + 2 \frac{dW}{dt} \]
\[ \frac{dP}{dt} = 2 \cdot 3 + 2 \cdot (-2) = 2 \text{ ft / min} \]

c. diagonal 
\[ r = \left( L^2 + W^2 \right)^{1/2} \]
\[ \frac{dr}{dt} = \frac{1}{2} \left( L^2 + W^2 \right)^{-1/2} \left( 2 \frac{dL}{dt} + 2 \frac{dW}{dt} \right) = \frac{1}{2} \left( (15)^2 + (40)^2 \right)^{-1/2} (2 \cdot 3 + 2 \cdot (-2)) \]
\[ = \frac{1}{\sqrt{1825}} \text{ ft / min} \]

5. The volume of a tree is given by 
\[ V = \frac{1}{12\pi} C^2 h \] where \( C \) is the circumference of the tree in meters at ground level and \( h \) is the height of the tree in meters. Both \( C \) and \( h \) are functions of time \( t \) in years.

a. Find a formula for \( \frac{dV}{dt} \). What does it represent in practical terms?
\[ \frac{dV}{dt} = \frac{1}{12\pi} \left( C^2 \frac{dh}{dt} + h \cdot 2C \frac{dC}{dt} \right) \]
\[ \Delta \text{ in circumference over time} \]

b. Suppose the circumference grows at a rate of 0.2 meters/year and the height grows at a rate of 4 meters/year. How fast is the volume of the tree growing when the circumference is 5 meters and the height is 22 meters?
\[ \frac{dV}{dt} = \frac{5}{12\pi} (5 \cdot 4 + 2 \cdot 22 \cdot 0.2) \]
\[ \frac{dV}{dt} = \frac{5}{12\pi} (28.8) \]
\[ \frac{dV}{dt} = \frac{12}{\pi} \text{ m}^3 \text{ / year} \]

6. a. When the radius of a spherical balloon is 10 cm, how fast is the volume of the balloon changing with respect to change in its radius?
\[ v = \frac{4}{3} \pi r^3 \]
\[ \frac{dv}{dt} = 4\pi r^2 \frac{dr}{dt} \]
\[ \frac{dv}{dt} = 4\pi (10)^2 \frac{dr}{dt} = 400\pi \frac{dr}{dt} \]

b. If the radius of the balloon is increasing by 0.5 cm/sec, at what rate is the air being blown into the balloon when the radius is 6 cm?
\[ v = \frac{4}{3} \pi r^3 \]
\[ \frac{dv}{dt} = 4\pi r^2 \frac{dr}{dt} \]
\[ \frac{dv}{dt} = 4\pi (6)^2 (0.5) = 72\pi \text{ cm}^3 \text{ / sec} \]
7. When hyperventilating, a person breathes in and out very rapidly. A spirogram is a machine that draws a graph of the volume of air in a person’s lungs as a function of time. During hyperventilation, the person’s spirogram trace might be represented by \( V = 3 - 0.05 \cos(200 \pi t) \) where \( V \) is the volume of air in liters in the lungs at time \( t \) minutes.

a. Sketch a graph of one period of this function.

b. What is the rate of flow of air in liters/minute? Sketch a graph of this function.

\[
\frac{dV}{dt} = 0.05 \sin(200 \pi t) \cdot 200 \pi = 10 \pi \sin(200 \pi t)
\]

c. Mark the following on each of the graphs above.

i) the interval when the person is breathing in

(0.005, 0.01)

ii) the interval when the person is breathing out

(0, 0.005)

iii) the time when the rate of flow of air is a maximum when the person is breathing in

0.0075 min