LOGISTIC GROWTH
A real-world problem from Example 1 in exponential growth:

Under favorable conditions, a single cell of the bacterium *Escherichia coli* divides into two about every 20 minutes.

If the same rate of division is maintained for **10 hours**, how many organisms will be produced from a single cell?

Solution: 10 hours = 30 20-minute periods

There will be \(2^{30} = 1,073,741,824\) bacteria after 10 hours.

A problem that seems just as reasonable:

Under favorable conditions, a single cell of the bacterium *Escherichia coli* divides into two about every 20 minutes.

If the same rate of division is maintained for **10 days**, how many organisms will be produced from a single cell?

Solution: 10 days = 720 20-minute periods

There will be \(2^{720} \approx 5.5 \times 10^{216}\) bacteria after 10 days.
Makes sense…

…until you consider that there are probably fewer than $10^{80}$ atoms in the entire universe.
Why didn’t they tell us the truth? Most of those classical “exponential growth” problems should have been “logistic growth” problems!
Problem with population growth model: $k > 0$

$k > 0 \quad t \to \infty, \quad P \to \infty$

$\to growth is unlimited$

*no natural enemies, no sharks, no diseases, no famine*

Usually the growth in the real world is not unlimited. Real-life populations do not increase forever. There is some limiting factor such as food, living space or waste disposal, parking space…

Natural maximum exists, such that the growth cannot occur beyond it.

There is a maximum population, or **carrying capacity**, $M$.

After a while, things that start off growing exponentially begin to compete for resources like food, water, money, and parking spaces. The growth begins to taper off as it approaches some **carrying capacity** of the system.
A more realistic model is the *logistic growth model* where growth rate is proportional to both the amount present \((P)\) and the carrying capacity that remains: \((M-P)\)

In this case, the growth rate is not only proportional to the current value, but also how far the current value is from the carrying capacity.

The equation then becomes:

\[
\frac{dP}{dt} = kP(M - P)
\]

We can solve this differential equation to find the *logistics growth model*. 
Logistics Differential Equation

\[ \frac{dP}{dt} = kP(M - P) \]

\[ \frac{1}{P(M - P)} dP = kdt \]

\[ \frac{1}{M} \left( \frac{1}{P} + \frac{1}{M - P} \right) dP = M kdt \]

\[ \ln P - \ln (M - P) = Mkt + \text{const} \]

\[ \frac{P}{M - P} = Mkt + \text{const} \]
Logistics Differential Equation

\[
\frac{P}{M - P} = e^{Mk + \text{const}}
\]

\[
\frac{M - P}{P} = e^{-Mkt - \text{const}}
\]

\[
\frac{M}{P} - 1 = e^{-Mkt - \text{const}}
\]

\[
\frac{M}{P} = 1 + e^{-Mkt - \text{const}}
\]

Let \( C = e^{-\text{const}} \)

\[
P = \frac{M}{1 + Ce^{-(Mk)t}}
\]

Logistics Growth Model
Example: Ten grizzly bears were introduced to a national park 10 years ago. There are 23 bears in the park at the present time. The park can support a maximum of 100 bears.

Assuming a logistic growth model, when will the bear population reach 50? 75? 100?

\[
P = \frac{M}{1 + Ce^{-(Mk)t}} \quad M = 100 \quad P_0 = 10 \quad P_{10} = 23
\]

\[
10 = \frac{100}{1 + C} \quad \Rightarrow C = 9
\]

\[
P = \frac{100}{1 + 9e^{-100kt}}
\]

\[
23 = \frac{100}{1 + 9e^{-1000k}} \quad 9e^{-1000k} = \frac{77}{23} \quad k = 0.00098891
\]

\[
P = \frac{100}{1 + 9e^{-0.1t}}
\]
We can graph this equation and use “trace” to find the solutions.

\[ P = \frac{100}{1 + 9e^{-0.1t}} \]

- \( y = 50 \) at 22 years
- \( y = 75 \) at 33 years
- \( y = 100 \) at 75 years
A certain rumor spreads through a community at the rate
\[
\frac{dy}{dt} = 2y(1-y),
\]
where \(y\) is the proportion of the community that has heard the rumor at time \(t\).

(a) What proportion of the community has heard the rumor when it is spreading the fastest?
(b) If at time \(t = 0\) ten percent of the people have heard the rumor, find \(y\) as a function of \(t\).
(c) At what time \(t\) is the rumor spreading the fastest?

This was the first logistic differential equation to appear in an AP FRQ.
The first part did not involve solving the differential equation.

A certain rumor spreads through a community at the rate

$$\frac{dy}{dt} = 2y(1 - y),$$

where $y$ is the proportion of the community that has heard the rumor at time $t$.

(a) What proportion of the community has heard the rumor when it is spreading the fastest?

The answer was the value of $y$ that maximized $2y(1 - y)$ … equivalent to finding the vertex of an upside-down parabola!

$$\frac{d^2y}{dt^2} = 2 \frac{dy}{dt} - 4y \frac{dy}{dt} = 0 \quad \rightarrow \quad y = \frac{1}{2} \quad \rightarrow \quad \frac{dy}{dt} = \frac{1}{2}$$

The max occurs at $y = \frac{1}{2}$ ($y' = \frac{1}{2}$).
(b) If at time $t = 0$ ten percent of the people have heard the rumor, find $y$ as a function of $t$.

\[
\frac{dy}{dt} = 2y(1 - y)
\]

\[
\int \frac{dy}{y(1 - y)} = \int 2\,dt
\]

\[
\int \left(\frac{1}{y} + \frac{1}{1 - y}\right)\,dy = 2t + \text{const}
\]

\[
\ln y - \ln(1 - y) = 2t + \text{const} \quad \text{Note: } 0 < y < 1
\]

\[
\frac{y}{1 - y} = e^{2t + \text{const}} \quad \rightarrow \quad \frac{1 - y}{y} = e^{-2t - \text{const}} \quad \rightarrow \quad \frac{1}{y} = 1 + Ce^{-2t}
\]

\[
y = \frac{1}{1 + Ce^{-2t}}
\]
(b) If at time $t = 0$ ten percent of the people have heard the rumor, find $y$ as a function of $t$.

$$y = \frac{1}{1 + Ce^{-2t}} \quad y(t = 0) = 10\% = 0.1$$

$$0.1 = \frac{1}{1 + C} \quad \rightarrow C = 9$$

$$y = \frac{1}{1 + 9e^{-2t}}$$

(c) At what time $t$ is the rumor spreading the fastest?

(a) $\frac{1}{2}$ of the community ($y = \frac{1}{2}$) has heard the rumor when it is spreading the fastest ($y' = \frac{1}{2}$)

$$\frac{1}{2} = \frac{1}{1 + 9e^{-2t}} \quad \rightarrow 1 + 9e^{-2t} = 2 \quad \rightarrow \quad -2t = \ln \frac{1}{9}$$

$$t = \ln 3$$
Exponential growth occurs when the growth rate is proportional to the size of the population.

\[ \frac{dP}{dt} = kP \quad P = Ce^{kt} \]

Logistic growth occurs when the growth rate slows as the population approaches a maximal sustainable population \( M \).

\[ \frac{dP}{dt} = kP(M - P) \]

\[ P = \frac{M}{1 + Ce^{-(Mk)t}} \]
Before solving the differential equation, it is useful to observe how much information can be gleaned from the differential equation itself.

\[ \frac{dP}{dt} = kP(M - P) \]

Maximum of the growth:

\[ \frac{d^2P}{dt^2} = kM \frac{dP}{dt} - 2kP \frac{dP}{dt} = 0 \]

\[ P = \frac{M}{2} \]

The maximum is always at \( M/2 \)

\[ \text{slope} = 0 \text{ when } P = M, \frac{dP}{dt} = 0 \rightarrow \text{no growth any more} \]

When \( P < M \), the growth rate is positive. When \( P > M \), the growth rate is negative. In either case, the population approaches \( M \) as \( t \) increases. We can see this in a slope field for the differential equation.
Now let’s look at one of the most infamous logistic problems of all:

2004 / BC-5

Suffice it to say that the BC students in 2004 were not amused.
A population is modeled by a function $P$ that satisfies the logistic differential equation

$$\frac{dP}{dt} = \frac{P}{5} \left(1 - \frac{P}{12}\right).$$

(a) If $P(0) = 3$, what is $\lim_{t \to \infty} P(t)$?

If $P(0) = 20$, what is $\lim_{t \to \infty} P(t)$?

(b) If $P(0) = 3$, for what value of $P$ is the population growing the fastest?

(c) A different population is modeled by a function $Y$ that satisfies the separable differential equation

$$\frac{dY}{dt} = \frac{Y}{5} \left(1 - \frac{t}{12}\right).$$

Find $Y(t)$ if $Y(0) = 3$.

(d) For the function $Y$ found in part (c), what is $\lim_{t \to \infty} Y(t)$?
What the students were supposed to do was to answer (a), (b), and (c) using their rich knowledge of logistic functions.

A population is modeled by a function $P$ that satisfies the logistic differential equation

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(a) If $P(0) = 3$, what is $\lim_{t \to \infty} P(t)$?

If $P(0) = 20$, what is $\lim_{t \to \infty} P(t)$?

(b) If $P(0) = 3$, for what value of $P$ is the population growing the fastest?

(a) $12$ ; $12$  (b) $12/2 = 6$
Then they had to solve a separable differential equation in (c). The equation was *not* logistic. In fact, part (d) was designed to let them savor the difference.

Sadly, many of them never made it past (a) and (b).

(c) A different population is modeled by a function $Y$ that satisfies the separable differential equation

$$\frac{dY}{dt} = \frac{Y}{5} \left(1 - \frac{t}{12}\right).$$

Find $Y(t)$ if $Y(0) = 3$.

(d) For the function $Y$ found in part (c), what is $\lim_{t \to \infty} Y(t)$?
The logistic equation was:

\[
\frac{dP}{dt} = \frac{P}{5} \left( 1 - \frac{P}{12} \right)
\]

The equation in part (c) was:

\[
\frac{dy}{dt} = \frac{P}{5} \left( 1 - \frac{t}{12} \right)
\]

Notice that this population would grow until \( t = 12 \) and then crash fast as \( t \) got bigger and bigger.
Here’s the slope field for

\[ \frac{dy}{dt} = \frac{P}{5} \left( 1 - \frac{t}{12} \right) \]
And here’s the solution to the initial value problem in (c):

\[ \frac{dy}{dt} = \frac{y}{5} \left(1 - \frac{t}{12}\right) \]

\[ \frac{dy}{y} = \left(\frac{1}{5} - \frac{t}{60}\right) dt \]

\[ \ln y = \frac{t}{5} - \frac{t^2}{120} + C \]

\[ y = e^{\frac{t}{5} - \frac{t^2}{120} + C} \]

\[ y = Ae^{\frac{t}{5} - \frac{t^2}{120}} \]

\[ y(0) = 3 \Rightarrow A = 3 \]

\[ y = 3e^{\frac{t}{5} - \frac{t^2}{120}} \]
Then, finally, the limit in (d):

(d) For the function $Y$ found in part (c), what is $\lim_{t \to \infty} Y(t)$?

$$\lim_{t \to \infty} \left( 3e^5 - \frac{t^2}{120} \right) = 0.$$
As with the rumor problem, solving a logistic differential equation requires the use of partial fractions.

In fact, it can be done in the general case:

\[
\frac{dP}{dt} = kP(M - P)
\]

\[
P = \frac{M}{1 + Ae^{-(Mk)t}}
\]
A population is modeled by a function $P$ that satisfies the logistic differential equation

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Find $Y(t)$ if $Y(0) = 3$.

(d) For the function $Y$ found in part (c), what is $\lim_{t \to \infty} Y(t)$?
In the case of 2004 / BC-5:

\[
\frac{dP}{dt} = \frac{P}{5} \left(1 - \frac{P}{12}\right)
\]

\[
\frac{dP}{dt} = \frac{1}{60} P (12 - P)
\]

\[
\therefore P = \frac{12}{1 + Ae^{-(1/5)t}}
\]

\[
P = \frac{12}{1 + 3e^{-0.2t}} \quad \text{if } P(0) = 3.
\]
The Infection Game is a great way to show students that logistic growth is a real phenomenon.

Give everyone in the class a number. (For small classes, make it a hand infection and give each hand a number.)

Enter this command and “infect” a random victim. From now on, each new ENTER command will spread the infection.

Press ENTER again. You now have two infected victims at time $t = 2$.

Press ENTER two times (since each of the two infected victims can spread the infection.)

Now press ENTER 4 times…and so on.
The infection spreads exponentially at first, because you press ENTER more often as it spreads. But soon it slows down because there are fewer new victims to infect.

It actually gets exciting when there are two healthy students left and they wonder who will go first and how many ENTERs they can survive.

I gathered some data from an actual class…

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EDIT TESTS
8:LinReg(a+bx)
9:LnReg
0:ExpReg
A:PwrReg
3:Logistic
C:SinReg
D:Manual-Fit

Logistic L₁, L₂, Y
1